Roll No. .... Exam Code : J-22

Subject Code—55486

B. Sc. EXAMINATION

(Main/Reappear) (Batch 2018 Onwards)

(Third Semester)

MATHEMATICS

CML-306 Course–V

Advanced Calculus

Time : 3 Hours *Maximum Marks* : 80

Note : Attempt *Five* questions in all. Q. No. **1** is compulsory. Attempt *one* question from each Section.

(Compulsory Question)

1. (a) Prove that, a function f, which is continuous on a closed interval $[a, b]$ assumes every value between its bounds. **2½**

(b) If $y = \frac{3-u}{2+u}$ and $u = \frac{4x}{1-x^2}$; find $\frac{dy}{dx}$. 2½

(c) If $u = \sqrt{x^2 - y^2}\sin^{-1}\frac{y}{x}$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = u.$$ 2½

(d) Find $\frac{du}{dt}$, when $u = x^2 + y^2$, $x = at^2$, $y = 2at$. 2½

(e) Evaluate the indeterminate form :

$$\underset{x \to 0}{\mathrm{Lim}} + \frac{\log \sin x}{\cot x}.$$ 2

(f) Evaluate the integral in terms of Beta function $\int_0^1 x^3 \left(1-x^2\right)^{3/2} dx$. 2

(g) Evaluate the integral :

$\iint\limits_A x^2 y^3 dy\, dx$, where A is the rectangle $0 \le x \le 1,\ 0 \le y \le 3$. 2

Section I

2. (a) Show that the function f defined by $f(x)=\frac{x}{x+1}$ is uniformly continuous on $[0, 2]$. 8

(b) Verify Rolle's theorem for the function $f(x)=(x^2-4x+3)e^{2x}$ in $[1, 3]$. 8

3. (a) Verify Lagrange's mean value theorem for $f(x)=\cos x$ in $\left[0, \frac{\pi}{2}\right]$. 8

(b) Show that :

$$\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}=-\frac{e}{2}.$$ 8

Section II

4. (a) Let $f:\mathrm{R}^2 \to \mathrm{R}$ be continuous function. Define a function $g:\mathrm{R}^2 \to \mathrm{R}$ as :

$$g(x, y)=\begin{cases} f(x, y), & (x, y)\neq(0, 0) \\ f(x, y)+1, & (x, y)=(0, 0) \end{cases}.$$

Show that g is not continuous at $(0, 0)$. 8

(b) If $x^2+y^2+z^2=\frac{1}{u^2}$, prove that :

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0.$$ 8

5. (a) If u and v are functions of x and y defined by $x=u+e^{-v}\sin u$ and $y=v+e^{-v}\cos u$, prove that $\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$. 8

(b) Prove that :

$$y^x=1+2(y-1)+(x-2)(y-1)+(y-1)^2+\ldots$$ 8

Section III

6. State Young's theorem. Show that for the function :

$$f(x, y)=\begin{cases}\frac{x^2y^2}{x^2+y^2}, & (x, y)\neq(0, 0)\\ 0, & (x, y)=(0, 0)\end{cases}$$

$f_{xy}(0,0) = f_{yx}(0,0)$, even though the conditions of Young's theorem are not satisfied. 16

7. (a) Examine for maximum and minimum values the function :

$$\sin x + \sin y + \sin(x+y).$$ 8

(b) Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube. 8

Section IV

8. (a) Show that :

$$\int_0^\infty \frac{x^a}{a^x}\,dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}$$

if $a > 1$. 8

(b) Evaluate the integral $\int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\,dx\,dy$

and sketch the region of integration. 8

9. (a) Show that the volume of the tetrahedron bounded by the co-ordinate planes and the plane $x+y+z=1$ is $\frac{1}{6}$. **8**

(b) Change the order of integration and evaluate $\int_0^1 \int_{4y}^4 e^{x^2}\, dx\, dy$. **8**