Roll No. [redacted]

Exam Code : D-23

Subject Code—60485

B. Sc. EXAMINATION

(Main/Reappear) (Batch 2019 Onwards)

(Fifth Semester)

MATHEMATICS

CML-506(i), Paper–I

Groups and Rings

Time : 3 Hours *Maximum Marks :* 80

Note : Attempt *Five* questions in all, selecting *one* question from each Section. Q. No. 1 contains seven subquestions without any internal choice and is compulsory. All questions carry equal marks.

(Compulsory Question)

1\. (a) How many generators are there of the cyclic group of order 18 ? **3**

(b) If M and N are two normal subgroups of G such that $N \cap M = \{e\}$, then for every $n \in N$ and $m \in M$. Prove that $nm = mn$. 3

(c) Let 'a' be a fixed element of a group G. Then prove that the mapping $T_a : G \to G$ defined by $T_a(x) = a^{-1}xa$ is an automorphism of G. 2

(d) Prove that the centre of a ring R is a subring of R. 2

(e) Prove that the left annihilator of a left ideal is an ideal. 2

(f) Prove that an ideal of a ring of integers is maximal if and only if it is generated by some prime integer. 2

(g) Find all the ideals of the ring of real numbers. 2

Section I

2. (a) Show that an infinite cyclic group has exactly two generators. 8

(b) Show that every permutation can be written as product of disjoint cycles. 8

3. (a) Prove that the necessary and sufficient condition for a non-empty subset H of a group G to be a subgroup of G is that $ab^{-1} \in H$ for all $a, b \in H$. 8

(b) State and prove Cayley's theorem. 8

Section II

4. (a) Prove that the order of a subgroup of a finite group is the divisor of the order of the group. 8

(b) Show that every homomorphic image of a group G is isomorphic to some quotient group of G. 8

5. (a) Derive class equation for a finite group G. 8

(b) If N is a normal subgroup of G and $N \cap G' = \{e\}$, then prove that $N \subseteq Z(G)$, where G' denotes the commutator subgroup of G. 8

Section III

6. (a) Show that the order of each non-zero element of an integral domain (regarding the elements as the members of additive group) is same. 8

   (b) Let R be a commutative ring with unity such that every ideal of R is a prime ideal. Prove that R is a field. 8

7. (a) Let f be a ring isomorphism of R onto R′. Show that if R′ is an integral domain, then so is R. 8

   (b) Find the field of quotients of the integral domain $Z[i] = \{a + ib \mid a, b \in Z\}$. 8

Section IV

8. (a) Prove that the ring of integers is a principal ideal ring. 8

   (b) Show that the polynomial $8x^3 - 6x - 1$ is irreducible over Q. 8

9. (a) Let $f(x)$ and $g(x)$ be two non-zero polynomials of R [x], where R is a ring. Then prove the following : 8

   (i) $\deg(f(x).\ g(x)) \le \deg(f(x)) + \deg(g(x))$.

   (ii) If R is an integral domain, then $\deg(f(x).\ g(x)) = \deg(f(x)) + \deg(g(x))$, where $\deg(f(x))$ denotes the degree of the polynomial $f(x)$.

   (b) Prove that every principal ideal domain is a unique factorization domain. 8