Roll No. 2020423+ Exam Code : M-23

Subject Code—59294

B.Sc. EXAMINATION

(Main/Reappear)

(Batch 2018 Onwards)

(Sixth Semester)

MATHEMATICS

~~GMB-605-(i)~~

Linear Algebra

Time : 3 Hours *Maximum Marks* : 80

Note : Attempt *Five* questions in all, selecting *one* question from each Unit. Q. No. 1 is compulsory. All questions carry equal marks.

1. (a) Prove that union of two subspaces may not be a subspace. 3

(b) If u, v, w are linearly independent in V(F), where F is any subfield of C, then prove that the vectors $u + v$, $v + w$, $w + u$ are also linearly independent, where C is the set of complex numbers. 3

(c) Show that the function $T : R^2 \to R^2$ defined by $T(x, y) = (x - y, x + y)$ for $(x, y) \in R^2$ is bijective. 2

(d) Let T and S be two linear transformation from $R^3 \to R^2$ defined by $T(x, y, z) = (2x - 3y, 7y + 2z)$ and $S(x, y, z) = (x - z, y)$. Find the formula for defining the transformation 2S + 3T. 2

(e) Let $T : R^3 \to R^3$ be a linear transformation defined by $T(x, y, z) = (2x - y, x + y + z, 2z)$. Find the characteristic and minimal polynomial for T with respect to standard basis of R^3. 2

(f) If the vectors $u_1 = (1, 2i, i)$, $u_2 = (0, 1 + i, 1)$, $u_3 = (2, 1 - i, i) \in C^3$, then find the norm of each vector u_i. 2

(g) Define Adjoint operator. 2

Unit I

2. (a) Prove that a necessary and sufficient condition for a non-empty subset W of a vector space V(F) to be a subspace of V is that W is closed under addition and scalar multiplications. 8

(b) Determine the basis and dimension of the subspace W generated by the vectors (1, –1, 1), (8, 4, 2) (2, 2, 0), (3, 9, –3) of R^3. Also extend this basis to get a basis of R^3. 8

3. (a) If W_1 and W_2 are two subspaces of a finite dimensional vector space V(F), then prove that : 8

$$\dim (W_1 + W_2) = \dim W_1 + \dim W_2 - \dim (W_1 \cap W_2).$$

(b) Examine whether the following set of vectors $u = (y_1, y_2, y_3, y_4) \in R^4$ are subspaces of R^4 ? 8

(i) All u such that $y_1 + 2y_2 = 3y_3$.

(ii) All u such that $y_1 + y_2 + y_3 + y_4 = k$, a fixed real number.

Unit II

4\. (a) Prove that every n-dimensional vector space V(F) is isomorphic to F^n. 8

(b) If $T : R^4 \to R^3$ is a linear transformation defined by $T(1, 0, 0, 0) = (1, 1, 1)$; $T(0, 1, 0, 0) = (1, -1, 1)$, $T(0, 0, 1, 0) = (1, 0, 0)$; $T(0, 0, 0, 1) = (1, 0, 1)$; verify that Rank T + Nullity T = 4. 8

5\. (a) State and prove Rank and Nullity Theorem. 8

(b) Find a linear transformation which transforms the vectors $(1, 1, 1), (-1, 1, -1), (1, 1, 2)$ in R^3 to $(1, 1), (1, 1), (1, 0)$ in R^2. 8

Unit III

6. (a) Let $T : R^3 \to R^3$ be a linear transformation defined by $T(x, y, z) = (2x + y, y - z, 2y + 4z)$. Find the eigen values and eigen vectors with respect to the standard basis of R^3. 8

(b) Show that $T : R^3 \to R^3$ defined by $T(x, y, z) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta, z)$ is non-singular for all values of θ. 8

7. (a) Prove that the characteristic polynomial and the minimal polynomial of an operator T have same irreducible factors. 8

(b) Let $T : R^3 \to R^3$ be a linear transformation defined by $T(x, y, z) = (2y + z, x - 4y, 3x)$. Find the matrix of T with respect to the basis $\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$. Also verify that $[T, B][u, B] = [T(u), B]$ for all $u \in R^3$. 8

Unit IV

8. (a) Let $u = (a_1, a_2, \ldots, a_n)$ and $v = (b_1, b_2, \ldots, b_n)$ be arbitrary members of C^n. Define $<u, v> = a_1\overline{b_1} + a_2\overline{b_2} + \ldots + a_n\overline{b_n}$. Show that C^n is an inner product space. 8

(b) Using Gram-Schmidt orthogonalization process, construct an orthonormal basis of $V_3(C)$, given the basis $u_1 = (1 + i, i, 1)$, $u_2 = (2, 1 - 2i, 2 + i)$, $u_3 = (1 - i, 0, - i)$. 8

9. (a) Let α, β be two vectors in a complex inner product space. Then show that $\alpha \perp \beta$ if and only if $\| a\alpha + b\beta \|^2 = \| a\alpha \|^2 + \| b\beta \|^2$, where $a, b \in C$ are scalars. 8

(b) Let T be a linear operator on an inner product space V(F). Then prove that T^* exists and $TT^* = T^*T = 1$ if and only if T is unitary. 8