CONTENTS

1

Vector Spaces and Subspaces

EXERCISE 1.1

Example 1. **Prove that**

(i) C is a vector space over C
(ii) C is a vector space over R
(iii) Q is not a vector space over R
(iv) R is not a vector space over C
under usual operations of addition and scalar multiplication.

Solution. (*i*) Since set C of all complex number is a field and every field is a vector space over itself, therefore **C** is a vector space over **C**.

(*ii*) Since set **C** of complex number is an abelian group.

Let $a, b \in$ **R** and $z, z_1, z_2 \in$ **C**. Then $az \in$ **C** and

(*a*) $a(z_1 + z_2) = az_1 + az_2$

(*b*) $(a + b)z = az + bz$

(*c*) $a(bz) = (ab)z$.

(*d*) $1 \cdot z = z$.

Hence **C** is a vector space over **R**.

(*iii*) **Q** is not a vector space over **R**.

Note that **Q** is not closed with respect to scalar multiplication.

If $\sqrt{2} \in$ **R** and $3 \in$ **Q**, then $3\sqrt{2} \notin$ **Q** [*i.e.*, $a \in F, u \in V \not\Rightarrow au \in V$]

(*iv*) **R** is not a vector space over **C**.

Note that **R** is not closed with respect to scalar multiplication.

If $2 \in$ R, $3 + 4i \in$ C, then $2(3 + 4i) \notin$ R.

Example 2. **Which of the following sets form vector spaces over reals ?**

(*i*) All polynomials over R with constant term zero.
(*ii*) All polynomials over R with constant term 1.
(*iii*) Set of all ordered pairs (a, b) of integers.
(*iv*) All polynomials with positive coefficients.

Solution. (*i*) Let $V = \{f(x) : f(x) = a_1x + a_2x^2 + \ldots + a_nx^n, n \geq 1, a_i \in R\}$

Let $\quad f_1(x) = \sum_{i=1}^{m} a_i x^i, \; f_2(x) = \sum_{j=1}^{n} b_j x_j \in V$

$\therefore \quad f_1(x) + f_2(x) = \sum_{k=1}^{r} c_k x^k$, where $r = \max(m, n)$

and $c_k = a_k + b_k$

$\Rightarrow$ The addition composition defined in V is an internal binary composition in V

Also $\quad a\, f(x) = a \sum_{i=1}^{n} a_i x^i = \sum_{i=1}^{n} a a_i x^i = \sum_{i=1}^{n} b_i x^i \quad$ where $b_i = a a_i$

Thus scalar multiplication defined in V is an external binary composition in V.

All other properties for a vector space are obviously satisfied.

Hence V is a vector space over **R**.

(*ii*) Let $V = \{f(x) : f(x) = 1 + a_1 x + a_2 x^2 + \ldots + a_n x^n, n \geq 1, a_i \in \mathbf{R}\}$

Obviously V is not closed with respect to addition, as

$$f_1(x) = 1 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

$$f_2(x) = 1 + b_1 x + b_2 x^2 + \ldots + b_n x^n \in V$$

But $\quad f_1(x) + f_2(x) = 2 + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots \notin V$

Hence V is not a vector space over **R**

(*iii*) Let $V = \{(a, b) : a, b \in \mathbf{Z}\}$

Note that V is not closed with respect to scalar multiplication.

For example, $\quad (1, 2) \in V$ and $\frac{1}{3} \in \mathbf{R}$

whereas $\frac{1}{3}(1, 2) = \left(\frac{1}{3}, \frac{2}{3}\right) \notin V$

Hence V is not a vector space.

(*iv*) Let $V = \{f(x) : f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n, a_i \in \mathbf{R}^+\}$

Note that if $2 + 3x \in V$ and $-4 \in \mathbf{R}$,

then $\quad -4(2 + 3x) = -8 - 12x \notin V$

Hence V is not a vector space.

***Example* 3. Let $\mathbf{R}^+$ be the set of all positive real numbers. Define the operations of addition and scalar multiplication as follows :**

$$u + v = u.v, \text{ for all } u, v \in \mathbf{R}^+$$

$$au = u^a, \text{ for all } u \in \mathbf{R}^+ \text{ and } a \in \mathbf{R}.$$

Show that $\mathbf{R}^+$ is a vector space over R.

Solution. Obviously $\quad \mathbf{R}^+ \neq \phi$

Since product of two positive real numbers is also a positive real number, therefore $\mathbf{R}^+$ is closed with respect to addition.

I. Properties of Addition

(*i*) **Associativity :**

For all $u, v, w \in \mathbf{R}^+$, we have

$$(u + v) + w = (u \,.\, v) \,.\, w$$

and $\quad u + (v + w) = u \,.\, (v \,.\, w)$

Since $\quad (u \,.\, v) \,.\, w = u \,.\, (v \,.\, w)$

$\therefore \quad (u + v) + w = u + (v + w).$

(*ii*) **Commutativity :**

We have $\quad u + v = u \,.\, v \quad$ and $\quad v + u = v \,.\, u,$

Since $\quad u \,.\, v = v \,.\, u$

$\therefore \quad u + v = v + u.$

(*iii*) **Existence of Identity :**

There exists $1 \in \mathbf{R}^+$ such that

$$u + 1 = u.1 = u$$

and $\quad 1 + u = 1.u = u$

Thus 1 is the identity of $\mathbf{R}^+$.

(*iv*) **Existence of Inverse :**

For each $u \in \mathbf{R}^+$, there exists $\frac{1}{u} \in \mathbf{R}^+$ such that $u + \frac{1}{u} = u.\frac{1}{u} = 1.$

Moreover $\quad \frac{1}{u} + u = \frac{1}{u}.u = 1$

$\therefore \quad \frac{1}{u}$ is the inverse of u.

Thus $\mathbf{R}^+$ is an abelian group with respect to '+'.

II. Properties of Scalar Multiplication

The scalar multiplication is defined as

$$au = u^a \text{ for all } u \in \mathbf{R}^+ \text{ and } a \in \mathbf{R}$$

Moreover, the following properties are satisfied :

(*v*) $\quad 1.u = u^1 = u$

(*vi*) $\quad (ab)u = u^{ab}$

$$a(bu) = a(u^b) = (u^b)^a = u^{ba} = u^{ab}$$

$\therefore \quad (ab)u = a(bu)$

(*vii*) $\quad a(u + v) = (u + v)^a = (u \,.\, v)^a = u^a \,.\, v^a$

Also, $\quad au + av = u^a + v^a = u^a \,.\, v^a$

$\therefore \quad a(u + v) = au + av, \forall\, a \in \mathbf{R}$ and $u, v \in \mathbf{R}^+$

(viii) $(a+b)(u) = u^{a+b}$

$= u^a \cdot u^b = u^a + u^b = au + bu$

Thus $\mathbf{R}^+$ satisfies all the properties of a vector space over **R**.

Example 4. Show that R is a vector space over the field Q of rationals where vector addition is defined by

$$u + v = u + v, \quad \text{for all } u, v \in \mathbf{R}$$

and scalar multiplication is defined by :

$$a \cdot u = au, \text{ where } a \in \mathbf{Q}, u \in \mathbf{R}$$

Solution. I. We know that $< \mathbf{R}, + >$ forms an abelian group.

II. Since product of a rational number and a real number is always a real number,

$\therefore$ $a \in \mathbf{Q}, u \in \mathbf{R} \Rightarrow a.u = au$ (by definition) $\in \mathbf{R}$.

Moreover the following properties are also satisfied :

(i) $1.u = u$ for all $u \in \mathbf{R}$

(ii) $a(bu) = (ab)u$ for all $a, b \in \mathbf{Q}$ and $u \in \mathbf{R}$

(iii) $a(u+v) = au + av$ for all $a \in \mathbf{Q}$ and $u, v \in \mathbf{R}$

(iv) $(a+b)u = au + bu$ for all $a, b \in \mathbf{Q}$ and $u \in \mathbf{R}$

Hence **R** is a vector space over **Q**.

***Example 5.* Which of the following subsets of $\mathbf{R}^4$ are vector spaces for co-ordinatewise addition and scalar multiplication ?**

The set of all vectors $(x_1, x_2, x_3, x_4) \in \mathbf{R}^4$ such that

(a) $x_1 = 0$ (b) $x_1 < 0$ (c) $x_2 > 0$

(d) $x_4 = 0$ (e) $x_3^2 \geq 0$ (f) $2x_1 + 3x_2 = 0$.

Solution. (a) Let $V = \{(x_1, x_2, x_3, x_4) : x_1 = 0, x_2, x_3, x_4 \in \mathbf{R}\}$

Let $u = (0, x_2, x_3, x_4), v = (0, y_2, y_3, y_4) \in V$

$\therefore$ $u + v = (0, x_2 + y_2, x_3 + y_3, x_4 + y_4)$

and $au = a(0, x_2, x_3, x_4) = (0, ax_2, ax_3, ax_4)$

for some $a \in \mathbf{R}$

Thus V is closed with respect to vector addition and scalar multiplication.

(I) Properties of Addition

(i) **Associativity :**

Let $u, v, w, \in V$ be arbitrary

Now $(u+v)+w = ((0, x_2, x_3, x_4) + (0, y_2, y_3, y_4)) + (0, z_2, z_3, z_4)$

$= (0, x_2, x_3, x_4) + ((0, y_2, y_3, y_4) + (0, z_2, z_3, z_4))$

$= u + (v + w)$

$\therefore$ $(u + v) + w = u + (v + w)$

(ii) **Commutativity :**

Let $u, v \in V$ be arbitrary

$$\begin{aligned} u + v &= (0, x_2, x_3, x_4) + (0, y_2, y_3, y_4) \\ &= (0+0, x_2 + y_2, x_3 + y_3, x_4 + y_4) \\ &= (0+0, y_2 + x_2, y_3 + x_3, y_4 + x_4) \\ &= (0, y_2, y_3, y_4) + (0, x_2, x_3, x_4) \\ &= v + u. \end{aligned}$$

$\therefore$ $u + v = v + u$

(iii) **Existence of Identity :**

For every vector $u = (0, x_2, x_3, x_4) \in V$, there exist $(0, 0, 0, 0) \in V$ such that

$$u + \mathbf{0} = (0, x_2, x_3, x_4) + (0, 0, 0, 0) = (0, x_2, x_3, x_4) = u$$

Similarly, $\mathbf{0} + u = u$

$\therefore$ $u + \mathbf{0} = u = \mathbf{0} + u$

(iv) **Existence of Inverse :**

For any $u = (0, x_2, x_3, x_4) \in V$, there exists $-u = (0, -x_2, -x_3, -x_4) \in V$ such that

$$u + (-u) = (0, x_2, x_3, x_4) + (0, -x_2, -x_3, -x_4) = (0, 0, 0, 0) = \mathbf{0}$$

Similarly, $(-u) + u = \mathbf{0}$

$\therefore$ $u + (-u) = \mathbf{0} = (-u) + u$.

(II) Properties of Scalar Multiplication

(v) For $u \in V$ and $1 \in \mathbf{R}$, we have

$1.u = 1.(0, x_2, x_3, x_4) = (0, x_2, x_3, x_4)$

$\therefore$ $1.u = u$ for all $u \in V$

(vi) $a(bu) = (ab)u$ for all $a, b \in \mathbf{R}$ and $u \in V$

(vii) For $u, v \in V$ and $a \in R$, we have

$$\begin{aligned} a(u+v) &= a((0, x_2, x_3, x_4) + (0, y_2, y_3, y_4)) \\ &= a(0, x_2 + y_2, x_3 + y_3, x_4 + y_4) \\ &= (0, ax_2 + ay_2, ax_3 + ay_3, ax_4 + ay_4) \\ &= (0, ax_2, ax_3, ax_4) + (0, ay_2, ay_3, ay_4) \\ &= a(0, x_2, x_3, x_3) + a(0, y_2, y_3, y_4) \\ &= au + av \end{aligned}$$

$\therefore$ $a(u+v) = au + av$

(viii) Similarly, $(a+b)u = au + bu$ for all $a, b \in \mathbf{R}$ and $u \in V$.

Thus V satisfies all the properties of a vector space and hence is a vector space.

(b) Let $V = \{(x_1, x_2, x_3, x_4) : x_i \in \mathbf{R}$ and $x_1 < 0\}$

Obviously V is not a vector space as identity element does not exist in V.

[$\because$ *For every vector $u \in V$, there does not exist $(0,0,0,0) \in V$ ($\because x_1 < 0$)*]

(c) Let $V = \{(x_1, x_2, x_3, x_4) : x_i \in \mathbf{R}$ and $x_2 > 0$

Obviously V is not a vector space as identity element does not exist in V.

(d) Proceed as in part (i)

(e) Let $V = \{(x_1, x_2, x_3, x_4) : x_i \in \mathbf{R}$ and $x_3^2 \geq 0\}$

V satisfies all the condition for a vector space (Please try yourself)

Hence V is a vector space.

(f) Let $V = \{(x_1, x_2, x_3, x_4) : x_i \in \mathbf{R}$ and $2x_1 + 3x_2 = 0\}$

Obviously $V \neq 0$ [$\because (-3, 2, 4, 0) \in V$]

(I) Properties of Addition

Let $u = (x_1, x_2, x_3, x_4)$, $v = (y_1, y_2, y_3, y_4)$, $w = (z_1, z_2, z_3, z_4) \in V$ be arbitrary

Then $2x_1 + 3x_2 = 0, 2y_1 + 3y_2 = 0, 2z_1 + 3z_2 = 0$

(i) **Associativity :**

$$u + (v + w) = (x_1, x_2, x_3, x_4) + ((y_1, y_2, y_3, y_4) + (z_1, z_2, z_3, z_4))$$
$$= ((x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4)) + (z_1, z_2, z_3, z_4)$$
$$= (u + v) + w$$

$\therefore \quad u + (v + w) = (u + v) + w$.

(ii) **Commutativity :**

$$u + v = (x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4)$$
$$= (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4)$$
$$= (y_1 + x_1, y_2 + x_2, y_3 + x_3, y_4 + x_4)$$

[$\because x_i, y_i \in$ *R and R is commutative*]

$$= (y_1, y_2, y_3, y_4) + (x_1, x_2, x_3, x_4)$$
$$= v + u$$

$\therefore \quad u + v = v + u$.

(iii) **Existence of Identity :**

For every vector $u \in V$, there exist $(0, 0, 0, 0) \in V$ such that

$$u + \mathbf{0} = (x_1, x_2, x_3, x_4) + (0, 0, 0, 0)$$
$$= (x_1, x_2, x_3, x_4)$$
$$= u$$

Similarly, $0 + u = u$

$\therefore \quad u + \mathbf{0} = u = \mathbf{0} + u$

(iv) **Existence of Inverse**

For every $u = (x_1, x_2, x_3, x_4) \in V$, there exist

$-u = (-x_1, -x_2, -x_3, -x_4) \in V$ such that

$$u + (-u) = \mathbf{0} = (-u) + u$$

II. Properties of Scalar Multiplication

(v) $1 \cdot u = u$ for all $u \in V$, $1 \in \mathbf{R}$.

(vi) $a(bu) = (ab)u$ for all $a, b \in \mathbf{R}$ and $u \in V$.

(vii) $a(u + v) = au + av$ for all $a \in \mathbf{R}$ and $u, v \in V$

(viii) $(a + b)u = au + bu$ for all $a, b \in \mathbf{R}$ and $u \in V$.

Hence V is a vector space over **R**.

***Example* 6. Consider the set $\mathbf{R}^n$ of all ordered n-tuples of real numbers defined by :**

$$\mathbf{R}^n = \{u = (x_1, x_2, \ldots, x_n) ; x_i \text{ is real}, i = 1, \ldots, n\}$$

Show that $\mathbf{R}^n$ is a vector space over R.

Solution. The given set is

$$\mathbf{R}^n = \{u = (x_1, x_2, \ldots, x_n) : x_i \text{ is real}, i = 1, 2, 3, \ldots, n\}$$

In $u = (x_1, x_2, \ldots, x_n)$

x_i is called the i^{th} component or co-ordinate of u.

Let $u = (x_1, x_2, \ldots, x_n)$

and $v = (y_1, y_2, \ldots, y_n) \in \mathbf{R}^n$ be arbitrary.

Then $u + v = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ is called vector addition and $au = (ax_1, ax_2, \ldots, ax_n)$ is called scalar multiplication, where a is a real number.

I. Properties of Addition

(i) **Associativity :**

$$(u + v) + w = u + (v + w) \text{ for all } u, v, w \in \mathbf{R}^n$$

(ii) **Commutativity :**

$$u + v = v + u \text{ for all } u, v, \in \mathbf{R}^n.$$

(iii) **Existence of Identity :**

For each $u \in \mathbf{R}^n$, there exist $\mathbf{0} = (0, 0, \ldots, 0) \in \mathbf{R}^n$ such that $u + \mathbf{0} = u = \mathbf{0} + u$

(iv) **Existence of Inverse :**

For each $u \in \mathbf{R}^n$, there exist $-u = (-x_1, -x_2, \ldots, -x_n) \in \mathbf{R}^n$ such that $u + (-u) = \mathbf{0} = (-u) + u$

II. Properties of Scalar Multiplication

(*v*) $1.u = u$ for all $u \in \mathbf{R}^n$ and $v \in \mathbf{R}$

(*vi*) $a(bu) = (ab)u$ for all $u \in \mathbf{R}^n$ and $a, b \in \mathbf{R}$

(*vii*) $a(u + v) = au + av$ for all $u, v \in \mathbf{R}^n$ and $a \in \mathbf{R}$

(*viii*) $(a + b)u = au + bu$ for all $u \in \mathbf{R}^n$ and $a, b \in \mathbf{R}$

Thus all the conditions for a vector space are satisfied and hence $\mathbf{R}^n$ is a vector space over **R**.

Example 7. **Prove that the set of all matrices of the form $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$, where $x, y \in C$, is a vector space over C with respect to matrix addition and scalar multiplication.**

Solution. Let $M = \left\{ \begin{bmatrix} x & y \\ -y & x \end{bmatrix} : x, y \in \mathbf{C} \right\}$

If $A = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}, B = \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix} \in M$ be arbitrary,

then $$A + B = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ -(y_1 + y_2) & x_1 + x_2 \end{bmatrix} \in M$$

Thus the addition is defined in M and is an internal composition in M.

Moreover, if $A = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$ and $a \in \mathbf{C}$, then

$$aA = \begin{bmatrix} ax & ay \\ -ay & ax \end{bmatrix} \in M$$

Thus, scalar multiplication is defined and is an external composition in M.

I. Properties of Addition

(*i*) **Associativity :**

Let $A = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}, B = \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix}$ and $C = \begin{bmatrix} x_3 & y_3 \\ -y_3 & x_3 \end{bmatrix} \in M$

$$\therefore \quad (A + B) + C = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ -y_1 - y_2 & x_1 + x_2 \end{bmatrix} + \begin{bmatrix} x_3 & y_3 \\ -y_3 & x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 + x_3 & y_1 + y_2 + y_3 \\ -y_1 - y_2 - y_3 & x_1 + x_2 + x_3 \end{bmatrix}$$

Similarly, $$A + (B + C) = \begin{bmatrix} x_1 + x_2 + x_3 & y_1 + y_2 + y_3 \\ -y_1 - y_2 - y_3 & x_1 + x_2 + x_3 \end{bmatrix}$$

$$\therefore \quad (A + B) + C = A + (B + C).$$

(*ii*) **Commutativity :**

Let $A = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}, B = \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix} \in M$

$$\therefore \quad A + B = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ -y_1 - y_2 & x_1 + x_2 \end{bmatrix} = \begin{bmatrix} x_2 + x_1 & y_2 + y_1 \\ -y_2 - y_1 & x_2 + x_1 \end{bmatrix}$$

$$= \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix} + \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} = B + A.$$

(*iii*) **E tence of Identity :**

There exists a unique matrix $\mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in M$ such that

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} x & y \\ -y & x \end{bmatrix} = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

(*iv*) **Existence of Inverse :**

For each $A = \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \in M$, there exist

$$-A = \begin{bmatrix} -x & -y \\ y & -x \end{bmatrix} \in M, \text{ such that}$$

$$\therefore \quad A + (-A) = \mathbf{O} = (-A) + A$$

II. Properties of Scalar Multiplication :

(*v*) $1.A = A$ for all $A \in M$ and $1 \in \mathbf{C}$

(*vi*) $a(bA) = (ab)A$, for all $a, b \in \mathbf{C}$ and $A \in M$

(*vii*) $a(A + B) = aA + aB$, for all $A, B \in M, a \in \mathbf{C}$

Note that $A = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix}, B = \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix}$

$$\Rightarrow \quad A + B = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ -y_1 - y_2 & x_1 + x_2 \end{bmatrix}$$

$$\therefore \quad a(A + B) = \begin{bmatrix} a(x_1 + x_2) & a(y_1 + y_2) \\ -a(y_1 + y_2) & a(x_1 + x_2) \end{bmatrix} = \begin{bmatrix} ax_1 + ax_2 & ay_1 + ay_2 \\ -ay_1 - ay_2 & ax_1 + ax_2 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 & ay_1 \\ -ay_1 & ax_1 \end{bmatrix} + \begin{bmatrix} ax_2 & ay_2 \\ -ay_2 & ax_2 \end{bmatrix} = aA + aB$$

(*viii*) Similarly, $(a+b)A = aA + bA$ for all $a, b \in \mathbf{C}$ and $A \in M$.

Thus M satisfies all the properties for a vector space and hence M is a vector space over **C**.

Example 8. **Prove that the set of all diagonal matrices over R, is a vector space with respect to matrix addition and scalar multiplication.**

Solution. Let $M = \{A : A$ is a diagonal matrix of order $n \times n\}$.

Since sum of two diagonal matrices of the same order is also a diagonal matrix of the same order, therefore M is closed with respect to addition.

Also, if $A \in M$ and $\lambda \in \mathbf{R}$ be arbitrary, then $\lambda A \in M$ which shows that M is closed with respect to scalar multiplication.

I. Properties of Addition :

$< M, + >$ forms an abelian group (verify yourself).

II. Properties of Scalar Multiplication :

(*i*) $1.A = A$ for all $A \in M$ and $1 \in \mathbf{R}$

(*ii*) $a(bA) = (ab)A$ for all $a, b \in \mathbf{R}$ and $A \in M$

(*iii*) $a(A + B) = aA + aB$ for all $a \in \mathbf{R}$ and $A, B \in M$

(*iv*) $(a+b)A = aA + bA$ for all $a, b \in \mathbf{R}$ and $A \in M$.

Hence M is a vector space over **R**.

Example 9. Let $\mathbf{V = \{(a, b) : a, b \in R\}}$ **and F = R. Show that V is not a vector space under addition and scalar multiplication defined as in each one of the following cases :**

(*i*) $(a, b) + (c, d) = (a + c, b + d)\ ;\ \alpha(a, b) = (0, \alpha b)$

(*ii*) $(a, b) + (c, d) = (0, b + d)\ ;\ \alpha(a, b) = (\alpha a, \alpha b)$

(*iii*) $(a, b) + (c, d) = (a + c, b + d)\ ;\ \alpha(a, b) = (\alpha a, b)$

Solution. The given set is

$$V = V = \{(a, b) : a, b \in \mathbf{R}\} \text{ and } F = \mathbf{R}$$

(*i*) Let $u = (a, b) \in V, a \neq 0$

$\therefore$ $1.u = (0, b) \neq u$

Thus V is not a vector space.

(*ii*) Since there exist no ordered pair $(e, f) \in V$ such that

$$(e, f) + (a, b) = (a, b) \text{ for all } (a, b) \in V$$

$< V, + >$ is not a group $[\because (e, f) + (a, b) = (0, b + f)$; *by defination*]

Thus V is not a vector space over **R**.

(*iii*) Let $u = (a, b) \in V$

$\therefore$ $(\alpha + \beta)u = ((\alpha + \beta)a, b)$, where $\alpha, \beta \in \mathbf{R}$ [*by definition*]

Also $\alpha u + \beta u = (\alpha a, b) + (\beta a, b) = ((\alpha + \beta)a, 2b)$

$\therefore$ $(\alpha + \beta)u \neq \alpha u + \beta u$

Hence V is not a vector space over **R**

EXERCISE 1.2

Example 1. **Let V be a vector space of $\mathbf{R}^3$. Examine, whether the following are subspaces of V or not :**

(*i*) $\mathbf{W = \{(x, y, z) : 3x + y - z = 0, x, y, z \in R\}}$

(*ii*) $\mathbf{W = \{(x, y, z) : x + y - z = 0, 2x + 3y - z = 0, x, y, z \in R\}}$

(*iii*) $\mathbf{W = \{(x_1, x_2, x_3) : x_1 + x_2 \geq 0; x_1, x_2, x_3 \in R\}}$

(*iv*) $\mathbf{W = \{(x_1, x_2, x_3) : x_1 x_2 = 0 ; x_1, x_2, x_3 \in R\}}$

Solution. (*i*) $W = \{(x, y, z) : 3x + y - z = 0, x, y, z \in \mathbf{R}\}$

Since $(1, -2, 1) \in W$, $W \neq \phi$

Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2) \in W$ and $a \in \mathbf{R}$ be arbitrary.

$$\therefore \quad au + v = a(x_1, y_1, z_1) + (x_2, y_2, z_2)$$
$$= (ax_1 + x_2, ay_1 + y_2, az_1 + z_2) \quad \ldots(1)$$

Also, $3x_1 + y_1 - z_1 = 0$

$3x_2 + y_2 - z_2 = 0$

$\Rightarrow$ $a(3x_1 + y_1 - z_1) + (3x_2 + y_2 - z_2) = 0$

$\Rightarrow$ $3ax_1 + 3x_2 + ay_1 + y_2 - (az_1 + z_2) = 0$

$\Rightarrow$ $3(ax_1 + x_2) + (ay_1 + y_2) - (az_1 + z_2) = 0 \quad \ldots(2)$

From (1) and (2), we have

$$au + v \in W$$

Thus, W is a subspace of $V(\mathbf{R}^3)$.

(*ii*) $W = \{(x, y, z) : x + y - z = 0, \ 2x + 3y - z = 0, \ x, y, z \in \mathbf{R}\}$

Since $(0, 0, 0) \in W$, $\therefore$ $W \neq \phi$

Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2) \in W$ be arbitrary

If $a \in \mathbf{R}$, then $au + v = a(x_1, y_1, z_1) + (x_2, y_2, z_2)$

$$= (ax_1 + x_2, ay_1 + y_2, az_1 + z_2)$$

Also, $x_1 + y_1 - z_1 = 0$; $2x_1 + 3y_1 - z_1 = 0$

and $x_2 + y_2 - z_2 = 0$; $2x_2 + 3y_2 - z_2 = 0$

$\Rightarrow \quad a(x_1 + y_1 - z_1) + (x_2 + y_2 - z_2) = 0$

and $\quad a(2x_1 + 3y_1 - z_1) + (2x_2 + 3y_2 - z_2) = 0$

$\Rightarrow \quad (ax_1 + x_2) + (ay_1 + y_2) - (az_1 + z_2) = 0$

and $\quad 2(ax_1 + x_2) + 3(ay_1 + y_2) - (az_1 + z_2) = 0$

$\Rightarrow \quad au + v \in W$

Thus, W is a subspace of V.

(*iii*) Here $W = \{(x_1, x_2, x_3) : x_1 + x_2 \geq 0;\ x_1, x_2, x_3 \in \mathbf{R}\}$

Since W is not closed with respect to scalar multiplication.

$\therefore$ W is not a subspace of V

Note that, $u = (3, -2, 1) \in W$

and for $a = -1$,

$$au = -1(3, -2, 1)$$

$$= (-3, 2, -1) \notin W \qquad [\because -3 + 2 = -1 < 0]$$

(*iv*) Here $W = \{(x_1, x_2, x_3) : x_1 x_2 = 0;\ x_1, x_2, x_3 \in \mathbf{R}\}$

W is not a subspace of V, as W is not closed with respect to vector addition

e.g., $u = (1, 0, 2),\ v = (0, 1, 2) \in W$

whereas $u + v = (1, 1, 4) \notin W$.

***Example 2.* Let V be a vector space over $\mathbf{R}^3$. Examine whether the following are sub-spaces of V or not :**

(*i*) $W = \{(x, y, z) : x, y, z \in \mathbf{R}$ and $x^2 + y^2 + z^2 \leq 1\}$

(*ii*) $W = \{(x, y, z) : x \geq 0, x, y, z \in \mathbf{R}\}$ [K.U. 2008]

Solution. (*i*) Here $W = \{(x, y, z) : x, y, z \in \mathbf{R}$ and $x^2 + y^2 + z^2 \leq 1\}$

Let $u = (x, y, z) \in W$ such that $x^2 + y^2 + z^2 \leq 1\}$

Take $a = 2 \in \mathbf{R}$

$\therefore \quad au = 2(x, y, z) = (2x, 2y, 2z) \notin W$

$[\because (2x)^2 + (2y)^2 + (2z)^2 = 4(x^2 + y^2 + z^2) \leq 1$ *does not hold*$]$

$\Rightarrow$ W is not closed with respect to scalar multiplication

Hence W is not a subspace of $V(\mathbf{R}^3)$

(*ii*) Here $W = \{(x, y, z) : x \geq 0, x, y, z \in \mathbf{R}\}$

Note that W is not closed with respect to scalar multiplication, as

$u = (1, 2, 3) \in W$

If $a = -4 \in \mathbf{R}$, then $au = (-4, -8, -12) \notin W$

$[\because -4 < 0$ *and* $x \geq 0$ *is not satisfied*$]$

Hence W is not a subspace of V.

***Example 3.* (*a*) Let $V = \mathbf{R}^3 = \{(x, y, z) : x, y, z \in \mathbf{R}$. Show that the following subsets of V are subspaces of V.**

(*i*) $W = \{(x, y, z) : x - 3y + 4z = 0, x, y, z \in \mathbf{R}\}$

(*ii*) $W = \{(x, 2y, 3z) : x, y, z \in \mathbf{R}\}$ [M.D.U. 2000]

(*iii*) $W = \{(x, x, x) : x \in \mathbf{R}\}$

(*iv*) $W = \{(x, y, z) : x, y, z \in \mathbf{Q}\}$.

(*b*) Let V be the vector space of all polynomials in x of degree ≤ 5 (including the zero polynomial) over the field Q. If $f(x) \in V$, explain which of the following are subspaces of V :

(*i*) $W = \{f(x) :$ coeff. of x^3 is zero$\}$

(*ii*) $W = \{f(x) : f(x)$ is a polynomial of degree ≤ 3 or $f(x)$ is a zero polynomial $\}$

(*iii*) $W = \{f(x) : f(3) = 0\}$

(*iv*) $W = \{f(x) : f(1) = 0, f(5) = 0\}$

Solution. (*a*) (*i*) Let $W = \{(x, y, z) : x - 3y + 4z = 0, x, y, z \in \mathbf{R}\}$

Let $\quad u = (x_1, y_1, z_1),\ v = (x_2, y_2, z_2) \in W$ and $a \in \mathbf{R}$

$\therefore \quad au + v = a(x_1, y_1, z_1) + (x_2, y_2, z_2)$

$$= (ax_1 + x_2, ay_1 + y_2, az_1 + z_2)$$

Also $\quad u, v \in W \Rightarrow x_1 - 3y_1 + 4z_1 = 0,$

$$x_2 - 3y_2 + 4z_2 = 0$$

$\Rightarrow \quad a(x_1 - 3y_1 + 4z_1) + (x_2 - 3y_2 + 4z_2) = 0$

$\Rightarrow \quad ax_1 + x_2 - 3(ay_1 + y_2) + 4(az_1 + z_2) = 0$

$\Rightarrow \quad au + v \in W$.

Hence W is a subspace of V.

(*ii*) Let $W = \{x, 2y, 3z) : x, y, z \in \mathbf{R}\}$

Since $(1, 2, 3) \in W$, $\quad \therefore \quad W \neq \phi$

Let $\quad u = (x_1, 2y_1, 3z_1),\ v = (x_2, 2y_2, 3z_2) \in W$

$\therefore \quad a \in \mathbf{R}$ and $u, v \in W$

$\Rightarrow \quad au + v = (ax_1 + x_2, 2(ay_1 + y_2), 3(az_1 + z_2)) \in W$

Thus W is a subspace of V.

(*iii*) Let $W = \{(x, x, x) : x \in \mathbf{R}\}$

Since $(1, 1, 1) \in W$, $\quad \therefore \quad W \neq \phi$

Let $\quad u = (x_1, x_1, x_1), v = (y_1, y_1, y_1) \in W$ and $a \in \mathbf{R}$ be arbitrary.

$\therefore \quad au + v = a(x_1, x_1, x_1) + (y_1, y_1, y_1)$

$$= (ax_1 + y_1, ax_1 + y_1, ax_1 + y_1) \in W \qquad [\because ax_1 + y_1 \in \mathbf{R}]$$

Thus W is a subspace of V.

(*iv*) Proceed as in above part (*iii*).

(*b*) (*i*) Here $W = \{f(x) : \text{coeff. of } x^3 \text{ is zero}\}$

For any $u, v \in W \Rightarrow u + v \in W$,

[$\because$ *coeff. of* x^3 *in* u *and* v *are zero so coeff. of* x^3 *in* $u + v$ *is zero.*]

and any $u \in W, a \in \mathbf{Q} \Rightarrow au \in W$

Thus W is a subspace of V.

(*ii*) Proceed as in part (*i*).

(*iii*) Here $W = \{f(x) : f(3) = 0\}$

Obviously $W \neq \phi$ [$\because f(x) = x - 3 \in W$]

Let $f(x),\ g(x) \in W$

$\therefore$ $f(3) = 0$ and $g(3) = 0$

$\Rightarrow$ $f(x) + g(x) \in W$

[$\because$ *At* $x = 3, f(x) + g(x) = f(3) + g(3) = 0 + 0 = 0$]

Also if $a \in \mathbf{Q}$, then

$a f(x) \in W$ [$\because$ *At* $x = 3, a f(x) = a \cdot 0 = 0$]

$\therefore$ W is a subspace of V.

(*iv*) Here $W = \{f(x) : f(1) = 0, f(5) = 0\}$

Let $f(x),\ g(x) \in W$

$\therefore$ $f(1) = 0 = f(5)$,

and $g(1) = 0 = g(5)$

$\Rightarrow$ $f(x) + g(x) \in W$ [$\because$ *At* $x = 1$ *or* $x = 5, f(x) = 0, g(x) = 0$]

Also, if $a \in \mathbf{Q}$, then

$a f(x) \in W$ [$\because$ *At* $x = 1$ *or* $x = 5, f(x) = 0$

$\Rightarrow a f(x) = a \cdot 0 = 0$]

$\therefore$ W is a subspace of V.

Example 4. **If V is vector space of all real valued functions of real variable defined on [0, 1], show that the following subsets of V are sub-spaces :**

(*i*) all even functions

(*ii*) all continuous functions

Solution. (*i*) Let $W = \{f : f \in V \text{ and } f \text{ is even}\}$

If $f, g \in W$, then

$$f(-x) = f(x), \quad g(-x) = g(x), \quad \text{for all } x \in \mathbf{R}$$

Now $(a f + g)(x) = a f(x) + g(x)$ where $a \in \mathbf{R}$ be arbitrary

Also, $(a f + g)(-x) = a f(-x) + g(-x)$

$= a f(x) + g(x)$

$\Rightarrow$ $(af + g)$ is an even function and so belongs to W.

Hence W is a subspace of V.

(*ii*) Let $W = \{f : f \in V \text{ and } f \text{ is continuous}\}$

If $f, g \in W$, then for any $a \in \mathbf{R}$,

$af + g$ is also a continuous function and so belongs to W.

[$\because$ *Sum of two continuous functions is again a continuous function and product of a continuous function by a real number is also a continuous function*].

Example 5. **If V is the vector space of all *n*-square matrices over R, which of the following sets of matrices are subspaces of V ?**

(*i*) all lower triangular matrices

(*ii*) all scalar matrices

(*iii*) all symmetric matrices

(*iv*) all singular matrices

(*v*) all diagonal matrices

Solution. (*i*) Let $W = \{[a_{ij}]_{n \times n} : a_{ij} = 0 \text{ for } i < j, a_{ij} \in \mathbf{R}\}$ be the set of all lower triangular matrices.

Obviously $W \neq \phi$ [$\because I_n \in W$]

Let $A = [a_{ij}]_{n \times n},\ B = [b_{ij}]_{n \times n} \in W$ be arbitrary

Here $a_{ij} = 0,\ b_{ij} = 0,$ whenever $i < j$ ...(1)

Let $a \in \mathbf{R}$, then

$$aA + B = a[a_{ij}]_{n \times n} + [b_{ij}]_{n \times n} = [a\, a_{ij} + b_{ij}]_{n \times n} = [c_{ij}]_{n \times n}$$

where $c_{ij} = a\, a_{ij} + b_{ij}$

$= a(0) + 0,$ whenever $i < j$ [From (1)]

$= 0$

$\Rightarrow$ $[c_{ij}]_{n \times n}$ is a lower triangular matrix and so belongs to W

Hence, W is a subspace of V.

(*ii*) Let $W = \{[a_{ij}]_{n \times n} : a_{ij} = k, \text{ if } i = j \text{ and } a_{ij} = 0, \text{ if } i \neq j\}$

be a set of scalar matrices.

Obviously, $W \neq \phi$ [$\because I_n \in W$]

Let $A = [a_{ij}]_{n \times n},\ B = [b_{ij}]_{n \times n} \in W$ and $a \in \mathbf{R}$

$$\therefore \quad a_{ij} = \begin{cases} k_1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

$$\text{and} \quad b_{ij} = \begin{cases} k_2 & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Now $a\,A + B = a[a_{ij}] + [b_{ij}]$

$= [aa_{ij} + b_{ij}] = [c_{ij}]$

where $c_{ij} = a\,a_{ij} + b_{ij} = \begin{cases} ak_1 + k_2, & \text{if } i = j \\ a(0) + 0 = 0, & \text{if } i \neq j \end{cases}$

$\Rightarrow$ $[c_{ij}]$ is a scalar matrix.

$\Rightarrow$ $a\,A + B \in W$

Hence W is a subspace of V.

(*iii*) Let $W = \{A : A \text{ is a symmetric matrix}\}$

Let $A, B \in W$

$\Rightarrow$ $A' = A,\ B' = B$

Now if $a \in R$, then

$$(a\,A + B)' = a\,A' + B'$$
$$= a\,A + B \in W$$

Hence, W is a subspace of V.

(*iv*) W is not a subspace of V

Note that sum of two singular matrices need not be a singular matrix.

e.g., $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are both singular matrices but their sum $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not a singular matrix.

(*v*) Let $W = \{[a_{ij}]_{n \times n} : a_{ij} = 0, \text{ if } i \neq j, a_{ij} \in \mathbf{R}\}$ be a set of diagonal matrices.

Let $A = [a_{ij}],\ B = [b_{ij}] \in W$

If $a \in R$, then

$$a\,A + B = a[a_{ij}] + [b_{ij}] = [a\,a_{ij} + b_{ij}]$$
$$= [c_{ij}] \text{ where } c_{ij} = a\,a_{ij} + b_{ij}$$

But $a_{ij} = 0,\ \text{if } i \neq j$

and $b_{ij} = 0,\ \text{if } i \neq j$ $\quad [\because A, B \in W]$

$\therefore$ $c_{ij} = a(0) + 0,\ \text{if } i \neq j$

$= 0 \quad \text{if } i \neq j$

Thus $a\,A + B$ is a diagonal matrix and so belongs to W

Hence, W is a subspace of V.

***Example 6.* Show that $W = \left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} : x \in \mathbf{R}\right\}$ is a subspace of the vector space V of all 2 × 2 real matrices.**

Solution. Here $W = \left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} : x \in \mathbf{R}\right\}$

Let $A = \begin{bmatrix} x_1 & 0 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} x_2 & 0 \\ 0 & 0 \end{bmatrix} \in W$ be arbitrary.

If $a \in \mathbf{R}$, then $aA + B = a\begin{bmatrix} x_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} x_2 & 0 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} ax_1 + x_2 & 0 \\ 0 & 0 \end{bmatrix} \in W$$

Thus, W is a subspace of V.

***Example 7.* Show that $W = \{(x, y) : ax + by = 0;\ x, y \in \mathbf{R}\}$, where a, b are fixed real numbers, is a subspace of $\mathbf{R}^2$.**

Solution. Here $W = \{(x, y) : ax + by = 0; x, y \in \mathbf{R}\}$, where a, b are fixed real numbers.

Let $u = (x_1, y_1),\ v = (x_2, y_2) \in W$ be arbitrary

$\therefore$ $ax_1 + by_1 = 0$ and $ax_2 + by_2 = 0$ ...(1)

If $\alpha \in \mathbf{R}$ is arbitrary, then

$$\alpha u + v = \alpha(x_1, y_1) + (x_2, y_2)$$
$$= (\alpha x_1 + x_2,\ \alpha y_1 + y_2) \quad \text{...(2)}$$

Also, $\alpha\,(ax_1 + by_1) + (ax_2 + by_2) = 0$ [*By (1)*]

or $a\,(\alpha x_1 + x_2) + b\,(\alpha y_1 + y_2) = 0$

$\Rightarrow$ $\alpha u + v \in W$ [*By (2)*]

Hence W is a subspace of $\mathbf{R}^2$.

***Example 8.* Let R be the field of real numbers. Show that $\{a, b, c, d : b - 2c + d = 0\}$ is a proper sub-space of $\mathbf{R}^4$ over R.**

Solution. Here $W = \{(a, b, c, d) : b - 2c + d = 0\}$ ...(1)

Obviously $W \neq \phi$ $\quad [\because (0, 1, 1, 1) \in W]$

Let $u = (a_1, b_1, c_1, d_1),\ v = (a_2, b_2, c_2, d_2) \in W$

and $\alpha \in \mathbf{R}$ be arbitrary

$$\therefore \quad \alpha u + v = \alpha\,(a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2)$$
$$= (\alpha a_1 + a_2, \alpha b_1 + b_2, \alpha c_1 + c_2, \alpha d_1 + d_2) \quad \text{...(2)}$$

Also, $b_1 - 2c_1 + d_1 = 0$

$b_2 - 2c_2 + d_2 = 0$

$\Rightarrow$ $\alpha(b_1 - 2c_1 + d_1) + (b_2 - 2c_2 + d_2) = 0$

$\Rightarrow$ $\alpha b_1 + b_2 - 2(\alpha c_1 + c_2) + \alpha d_1 + d_2 = 0$ ...(3)

From (1), (2) and (3), it follows that $\alpha u + v \in W$

Hence W is a proper subspace of $\mathbf{R}^4$ over R

[$\because$ *{0} and W are improper (trivial) subspaces of* $\mathbf{R}^4$.]

EXERCISE 1.3

Example 1. **Define linear sum of two subspaces and show that linear sum of two subspaces is also a subspace of the vector space V(F).**

Solution. **Definition :** If W_1 and W_2 are two subspaces of a vector space V(F), then we define the linear sum of W_1 and W_2 as

$$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1 \text{ and } w_2 \in W_2\}.$$

Let W_1 and W_2 be two subspaces of a vector space V(F) then $W_1 + W_2$ be the linear sum of W_1 and W_2.

Let u, v be any two elements of $W_1 + W_2$. Then by definition

$$u = w_1 + w_2 \text{ for some } w_1 \in W_1, w_2 \in W_2$$

and $$v = w_1' + w_2', \text{ where } w_1' \in W_1, w_2' \in W_2$$

Now, for $a, b \in F$,

Consider, $au + bv = a(w_1 + w_2) + b(w_1' + w_2')$

$$= (aw_1 + bw_1') + (aw_2 + bw_2')$$

Since W_1 and W_2 are subspaces of V,

$\therefore$ $$aw_1 + bw_1' \in W_1 \text{ and } aw_2 + bw_2' \in W_2$$

Thus $$au + bv = (aw_1 + bw_1') + (aw_2 + bw_2') \in W_1 + W_2$$

i.e., $$au + bv \in W_1 + W_2$$

Hence, $W_1 + W_2$ is a subspace of V(F).

Example 2. **In the vector space $V = R^2(R)$; let $W_1 = \{(x, 0) : x \in R\}$ and $W_2 = \{(0, x) : x \in R\}$. Prove that W_1, W_2 are subspaces of $R^2(R)$ and show that $V = W_1 \oplus W_2$.**

Solution. Let $W_1 = \{(x, 0) : x \in \mathbf{R}\}$

and $$W_2 = \{(0, x) : x \in \mathbf{R}\}$$ [*verify*]

Let $w_1 = (x, 0)$ and $w_2 = \{(y, 0) \in W_1$ be arbitrary

If $\alpha, \beta, \in \mathbf{R}$ then $\alpha w_1 + \beta w_2 = \alpha(x, 0) + \beta(y, 0)$

$$= (\alpha x + \beta y, 0) = \{(x', 0), \text{ where } x' = \alpha x + \beta y \in R\}$$

$\therefore \alpha, \beta \in R$ and $w_1, w_2 \in W_1 \Rightarrow \alpha w_1 + \beta w_2 \in W_1$

Hence W_1 is a subspace of $\mathbf{R}^2(\mathbf{R})$

Similarly, W_2 is also a subspace of $\mathbf{R}^2(\mathbf{R})$.

Now, we have to show that $V = W_1 \oplus W_2$

For any $$v = (a, b) \in V,$$

$$(a, b) = (a, 0) + (0, b) \in W_1 + W_2$$

$\therefore$ $$V = W_1 + W_2$$

Now, let $w \in W_1 \cap W_2$

$\Rightarrow$ $w \in W_1$ and $w \in W_2$

$\Rightarrow$ $w = (x, 0) \in W_1$ and $w = (0, y) \in W_2$

$\Rightarrow$ $(x, 0) = (0, y)$

$\Rightarrow$ $x = 0, y = 0$

$\Rightarrow$ $w = (0, 0)$

Thus, $W_1 \cap W_2 = \{0\}$

Hence $\mathbf{V = W_1 \oplus W_2}$

Example 3. **Let $V = R^2(R)$ be a vector space. Let**

$$\mathbf{W_1 = \{(x, x) : x \in R\}, W_2 = \{(x, 0) : x \in R\}.}$$

Prove that $V = W_1 \oplus W_2$.

Solution. Let $W_1 = \{(x, x) : x \in \mathbf{R}\}$ and $W_2 = \{(x, 0) : x \in \mathbf{R}\}$

For any $v = (a, b) \in V$,

$$(a, b) = (b, b) + (a - b, 0) \in W_1 + W_2$$

[$\because$ $(b, b) \in W_1$ *and* $(a - b, 0) \in W_2$]

$\therefore$ $$V = W_1 + W_2$$

(*ii*) Now, let $w \in W_1 \cap W_2$

$\Rightarrow$ $w \in W_1$ and $w \in W_2$

$\Rightarrow$ $w = (x, x) \in W_1$ and $w = (x, 0) \in W_2$

$\Rightarrow$ $(x, x) = (x, 0)$

$\Rightarrow$ $x = 0 \Rightarrow w = (0, 0)$

Thus, $W_1 \cap W_2 = \{0\}$

Hence $\mathbf{V = W \oplus W_2}$.

Example 4. **Let V be the vector space of n-square matrices over a field R. Let W_1 and W_2 be the subspaces of symmetric and skew-symmetric matrices of order n respectively. Show that V is the direct sum of W_1 and W_2.**

Solution. Let W_1 and W_2 be the subspaces of symmetric and skew-symmetric matrices of order n respectively.

Let A be any $n \times n$ matrix over a field R.

For any $A \in V$,

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = P + Q \text{ (say)}$$

where $$P = \frac{1}{2}(A + A') \text{ and } Q = \frac{1}{2}(A - A')$$

Now, $\quad P' = \left[\frac{1}{2}(A + A')\right]' = \frac{1}{2}(A' + A) = P$

$\Rightarrow$ P is symmetric

$\therefore \quad P \in W_1$

Also $\quad Q' = \left[\frac{1}{2}(A - A')\right]' = -\frac{1}{2}(A - A') = -Q$

$\Rightarrow$ Q is skew-symmetric

$\therefore \quad Q \in W_2$

Thus, $\quad A = P + Q$, where $P \in W_1$, $Q \in W_2$

$\Rightarrow \quad V = W_1 + W_2$

Let B be any $n \times n$ matrix over a field R.

Then $\quad B \in W_1 \cap W_2$

$\Rightarrow \quad B \in W_1$ and $B \in W_2$

$\Rightarrow \quad B' = B$ and $B' = -B$

$\Rightarrow \quad B = -B$

$\Rightarrow \quad 2B = 0$, where 0 is a $n \times n$ null matrix

$\Rightarrow \quad B = 0$

Thus, $\quad W_1 \cap W_2 = \{0\}$

Hence $\quad V = W_1 \oplus W_2$

MISCELLANEOUS EXERCISE

***Example 1.* Let $V = \{f(x) : f(x)$ is a polynomial of degree $\leq n$ (n is a non-negative integer) or $f(x)$ is a zero polynomial$\}$. Show that V is a vector space under addition and scalar multiplication of polynomials.**

Solution. Let **R** be a field and $V = \{f(x) : f(x)$ is a polynomial of degree $\leq n$ or $f(x)$ is a zero polynomial$\}$.

Let $\quad f(x) = a_0 + a_1 x + \ldots.. + a_k x^k$.

and $\quad g(x) = b_0 + b_1 x + \ldots.. + b_k x^k$.

where both $f(x)$ and $g(x)$ are either zero polynomials or of degree $\leq n$.

Now, we define addition and scalar multiplication as follows :

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1) x + \ldots..$$

$$\alpha f(x) = (\alpha a_0) + (\alpha a_1) x + \ldots.., \alpha \in \mathbf{R}.$$

I. Properties of Addition

(*i*) **Associativity :**

Let $f(x)$, $g(x)$, $h(x) \in V$ be arbitrary

$\Rightarrow [f(x) + g(x)] + h(x) = [(a_0 + a_1 x + a_2 x^2 + \ldots) + (b_0 + b_1 x + b_2 x^2 + \ldots)] + (c_0 + c_1 x + c_2 x^2 + \ldots)$

$= [(a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + \ldots] + (c_0 + c_1 x + c_2 x^2 + \ldots)$

$= [(a_0 + b_0) + c_0] + [(a_1 + b_1) + c_1] x + [(a_2 + b_2) + c_2] x^2 + \ldots$

$= [a_0 + (b_0 + c_0)] + [a_1 + (b_1 + c_1)] x + [a_2 + (b_2 + c_2)] x^2 + \ldots$

[$\because$ *Addition is associative in R*]

$= (a_0 + a_1 x + a_2 x^2 + \ldots) + [(b_0 + c_0) + (b_1 + c_1) x + (b_2 + c_2) x^2 + \ldots]$

$= (a_0 + a_1 x + a_2 x^2 + \ldots) + [(b_0 + b_1 x + b_2 x^2 + \ldots) + (c_0 + c_1 x + c_2 x^2 + \ldots)]$

$= f(x) + [g(x) + h(x)]$

$\therefore \quad [f(x) + g(x)] + h(x) = f(x) + [g(x) + h(x)]$

(*ii*) **Commutativity :**

Let $f(x)$ and $g(x) \in V$ be arbitrary

$\Rightarrow \quad f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + \ldots..$

$= (b_0 + a_0) + (b_1 + a_1) x + (b_2 + a_2) x^2 + \ldots..$

[$\because$ *Addition is commutative in R*]

$= (b_0 + b_1 x + b_2 x^2 + \ldots..) + (a_0 + a_1 x + a_2 x^2 + \ldots..)$

$= g(x) + f(x)$

$\therefore \quad f(x) + g(x) = g(x) + f(x)$

(*iii*) **Existenc of Identity :**

For every $\quad f(x) = a_0 + a_1 x + a_2 x^2 + \ldots. + a_k x^k \in V$

there exists $\quad \mathbf{0} = 0 + 0 \,.\, x + 0 \,.\, x^2 + \ldots.. + 0 \,.\, x^k \in V$ such that

$f(x) + \mathbf{0} = (a_0 + a_1 x + a_2 x^2 + \ldots.. + a_k x^k) + (0 + 0 \,.\, x + 0 \,.\, x^2 + \ldots.. + 0 \,.\, x^k)$

$= f(x)$

Similarly, $\mathbf{0} + f(x) = f(x)$

$\therefore \quad f(x) + \mathbf{0} = f(x) = \mathbf{0} + f(x)$.

(*iv*) **Existence of Inverse :**

For any $\quad f(x) = a_0 + a_1 x + \ldots.. + a_k x^k \in V$, there exists

$-f(x) = (-a_0) + (-a_1) x + \ldots.. + (-a_k) x^k \in V$ such that

$f(x) + (-f(x)) = (a_0 + a_1 x + \ldots.. + a_k x^k) + [(-a_0) + (-a_1) x + \ldots.. + (-a_k) x^k] = \mathbf{0}$

Similarly, $(-f(x))+f(x)=0$

$\therefore \quad f(x)+(-f(x))=0=(-f(x))+f(x)$

II. Properties of Scalar Multiplication

(*v*) For $f \in V$ and $1 \in \mathbf{R}$, we have

$$(1\,.\,f)(x) = 1\,.\,f(x) = f(x)$$

$\therefore \quad 1\,.\,f = f$ for all $f \in V$.

(*vi*) For $f \in V$ and $a, b \in \mathbf{R}$, we have

$$\begin{aligned}[(ab)f](x) &= (ab)f(x)\\ &= a(b\,f(x))\\ &= a\,[(bf)(x)]\\ &= [a\,(bf)](x)\end{aligned}$$

$\therefore \quad (ab)f = a(bf).$

(*vii*) For $f, g \in V$ and $a \in \mathbf{R}$, we have

$$\begin{aligned}[a\,(f+g)]x = a[(f+g)(x)] &= a[f(x)+g(x)]\\ &= a\,f(x)+a\,g(x)\\ &= (a\,f)(x)+(ag)(x)\\ &= (a\,f+ag)(x)\end{aligned}$$

$\therefore \quad a\,(f+g) = af+ag$

(*iii*) For $f \in V$ and $a, b \in \mathbf{R}$, we have

$$\begin{aligned}[(a+b)f](x) &= (a+b)f(x)\\ &= a\,f(x)+b\,f(x)\\ &= (af)(x)+(bf)(x)\\ &= (af+bf)(x)\end{aligned}$$

$\therefore \quad (a+b)f = af+bf.$

Thus V satisfies all the properties of vector space and hence V is a vector space over **R**.

Example 2. **Let** $\mathbf{V} = \{a + b\sqrt{2} + c\sqrt[3]{3} : a, b, c \in Q\}$. **Show that V is a vector space under addition and scalar multiplication of real numbers.**

Solution. Here $V = \{a + b\sqrt{2} + c\sqrt[3]{3} : a, b, c \in Q\}$

Let $u, v \in V$ such that

$$u = a_1 + b_1\sqrt{2} + c_1\sqrt[3]{3}\,,\ v = a_2 + b_2\sqrt{2} + c_2\sqrt[3]{3}$$

where $a_1, b_1, c_1, a_2, b_2, c_2 \in Q$

We define addition and scalar multiplication as follows :

Let $u, v \in V$

Then $u + v = (a_1 + b_1\sqrt{2} + c_1\sqrt[3]{3}) + (a_2 + b_2\sqrt{2} + c_2\sqrt[3]{3})$

$= (a_1 + a_2) + (b_1 + b_2)\sqrt{2} + (c_1 + c_2)\sqrt[3]{3} \in V$

$[\because a_1, a_2, b_1, b_2, c_1, c_2 \in Q \Rightarrow a_1 + a_2, b_1 + b_2, c_1 + c_2 \in Q]$

For $\alpha \in Q$ and $u \in V$, we have

$$\alpha u = \alpha(a + b\sqrt{2} + c\sqrt[3]{3})$$

$$= (\alpha a) + (\alpha b)\sqrt{2} + (\alpha c)\sqrt[3]{3} \in V$$

$[\because \alpha, a, b, c \in Q \Rightarrow \alpha a, \alpha b, \alpha c \in Q]$

Now, let us prove all the axioms of vector space.

I. Properties of Addition

(*i*) **Associativity.** Let $u, v, w \in V$

Now, $(u+v)+w = [(a_1 + b_1\sqrt{2} + c_1\sqrt[3]{3}) + (a_2 + b_2\sqrt{2} + c_2\sqrt[3]{3})] + (a_3 + b_3\sqrt{2} + c_3\sqrt[3]{3})$

$= [(a_1 + a_2) + (b_1 + b_2)\sqrt{2} + (c_1 + c_2)\sqrt[3]{3}] + (a_3 + b_3\sqrt{2} + c_3\sqrt[3]{3})$

$= \{[(a_1 + a_2) + a_3] + [(b_1 + b_2) + b_3]\sqrt{2} + [(c_1 + c_2) + c_3]\sqrt[3]{3}\}$

$= \{[a_1 + (a_2 + a_3)] + [b_1 + (b_2 + b_3)]\sqrt{2} + [c_1 + (c_2 + c_3)]\sqrt[3]{3}\}$

$[\because a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in Q$ *and associative law holds in* $Q]$

$= (a_1 + b_1\sqrt{2} + c_1\sqrt[3]{3}) + [(a_2 + a_3) + (b_2 + b_3)\sqrt{2} + (c_2 + c_3)\sqrt[3]{3}]$

$= (a_1 + b_1\sqrt{2} + c_1\sqrt[3]{3}) + [(a_2 + b_2\sqrt{2} + c_2\sqrt[3]{3}) + (a_3 + b_3\sqrt{2} + c_3)\sqrt[3]{3}]$

$= u + (v + w)$

$\therefore \quad (u+v)+w = u+(v+w)$

(*ii*) **Commutativity.** Let $u, v \in V$

Now, $u + v = (a_1 + b_1\sqrt{2} + c_1\sqrt[3]{3}) + (a_2 + b_2\sqrt{2} + c_2\sqrt[3]{3})$

$= (a_1 + a_2) + (b_1 + b_2)\sqrt{2} + (c_1 + c_2)\sqrt[3]{3}$

$= (a_2 + a_1) + (b_2 + b_1)\sqrt{2} + (c_2 + c_1)\sqrt[3]{3}$

$[\because$ *Commutativity holds in* $Q]$

$= (a_2 + b_2\sqrt{2} + c_2\sqrt[3]{3}) + (a_1 + b_1\sqrt{2} + c_1\sqrt[3]{3}) = v + u$

$\therefore \quad u + v = v + u$

(*iii*) **Existence of identity.**

Let $\quad u = a + b\sqrt{2} + c\sqrt[3]{3}, a, b, c \in Q$

Define **0** such that

$$\mathbf{0} = 0 + 0\sqrt{2} + 0\sqrt[3]{3} \in V$$

Now, $u + \mathbf{0} = (a + b\sqrt{2} + c\sqrt[3]{3}) + (0 + 0\sqrt{2} + 0\sqrt[3]{3})$

$= (a+0) + (b+0)\sqrt{2} + (c+0)\sqrt[3]{3}$

$= a + b\sqrt{2} + c\sqrt[3]{3} = u$

Also $\mathbf{0} + u = (0 + 0\sqrt{2} + 0\sqrt[3]{3}) + (a + b\sqrt{2} + c\sqrt[3]{3})$

$= (0+a) + (0+b)\sqrt{2} + (0+c)\sqrt[3]{3}$

$= a + b\sqrt{2} + c\sqrt[3]{3} = u$

$\therefore$ $u + \mathbf{0} = u = \mathbf{0} + u$ for all $u \in V$.

$\Rightarrow$ $\mathbf{0} = 0 + 0\sqrt{2} + 0\sqrt[3]{3}$ is the additive identity in V.

(*iv*) **Existence of inverse.**

Let $u = a + b\sqrt{2} + c\sqrt[3]{3},\ a, b, c \in Q$

Then $-u = (-a) + (-b)\sqrt{2} + (-c)\sqrt[3]{3} \in V$ $[\because -a, -b, -c \in Q]$

Now, $u + (-u) = [a + b\sqrt{2} + c\sqrt[3]{3}] + [(-a) + (-b)\sqrt{2} + (-c)\sqrt[3]{3}]$

$= [a + (-a)] + [b + (-b)]\sqrt{2} + [c + (-c)]\sqrt[3]{3}$

$= 0 + 0\sqrt{2} + 0\sqrt[3]{3} = \mathbf{0}$

and $(-u) + u = [(-a) + (-b)\sqrt{2} + (-c)\sqrt[3]{3}] + [a + b\sqrt{2} + c\sqrt[3]{3}]$

$= \{[(-a) + a] + [(-b) + b]\sqrt{2} + [(-c) + c]\sqrt[3]{3}\}$

$= 0 + 0\sqrt{2} + 0\sqrt[3]{3} = \mathbf{0}$

$\therefore$ $u + (-u) = \mathbf{0} = (-u) + u$

$\Rightarrow -u = (-a) + (-b)\sqrt{2} + (-c)\sqrt[3]{3}$ is the additive inverse of u in V.

II. Properties of scalar multiplication.

(*v*) For $u \in V$, $1 \in Q$, we have

$1.u = 1.(a + b\sqrt{2} + c\sqrt[3]{3})$

$= (1.a) + (1.b)\sqrt{2} + (1.c)\sqrt[3]{3}$

$= a + b\sqrt{2} + c\sqrt[3]{3} = u.$

$\therefore$ $1.u = u$ for all $u \in V$

(*vii*) For $u, v \in V$ and $\alpha \in Q$, we have

$\alpha(u + v) = \alpha[(a_1 + b_1\sqrt{2} + c_1\sqrt[3]{3}) + (a_2 + b_2\sqrt{2} + c_2\sqrt[3]{3})]$

$= \alpha[(a_1 + a_2) + (b_1 + b_2)\sqrt{2} + (c_1 + c_2)\sqrt[3]{3}]$

$= [\alpha(a_1 + a_2)] + [\alpha(b_1 + b_2)]\sqrt{2} + [\alpha(c_1 + c_2)]\sqrt[3]{3}$

$= (\alpha a_1 + \alpha a_2) + (\alpha b_1 + \alpha b_2)\sqrt{2} + (\alpha c_1 + \alpha c_2)\sqrt[3]{3}$

$= (\alpha a_1 + \alpha b_1\sqrt{2} + \alpha c_1\sqrt[3]{3}) + (\alpha a_2 + \alpha b_2\sqrt{2} + \alpha c_2\sqrt[3]{3})$

$= \alpha(a_1 + b_1\sqrt{2} + c_1\sqrt[3]{3}) + \alpha(a_2 + b_2\sqrt{2} + c_2\sqrt[3]{3})$

$\therefore$ $\alpha(u + v) = \alpha u + \alpha v.$

(*viii*) For $\alpha, \beta \in Q$ and $u \in V$, we have

$(\alpha + \beta)u = (\alpha + \beta)[a + b\sqrt{2} + c\sqrt[3]{3}]$

$= (\alpha + \beta)a + (\alpha + \beta)b\sqrt{2} + (\alpha + \beta)c\sqrt[3]{3}$

$= (\alpha a + \beta a) + (\alpha b + \beta b)\sqrt{2} + (\alpha c + \beta c)\sqrt[3]{3}$

$= (\alpha a + \alpha b\sqrt{2} + \alpha c\sqrt[3]{3}) + (\beta a + \beta b\sqrt{2} + \beta c\sqrt[3]{3})$

$= \alpha(a + b\sqrt{2} + c\sqrt[3]{3}) + \beta(a + b\sqrt{2} + c\sqrt[3]{3})$

$\therefore$ $(\alpha + \beta)u = \alpha u + \beta u.$

(*vi*) For $\alpha, \beta \in Q$ and $u \in V$, we have

$(\alpha\beta)u = (\alpha\beta)[a + b\sqrt{2} + c\sqrt[3]{3}],$

$= [(\alpha\beta)a] + [(\alpha\beta)b]\sqrt{2} + [(\alpha\beta)c]\sqrt[3]{3}$

$= \alpha(\beta a) + \alpha(\beta b)\sqrt{2} + \alpha(\beta c)\sqrt[3]{3}$

$= \alpha[\beta a + \beta b\sqrt{2} + \beta c\sqrt[3]{3}]$

$= \alpha[\beta(a + b\sqrt{2} + c\sqrt[3]{3})] = \alpha(\beta u).$

Hence V is a vector space over Q.

Example 3. **Let $V = \{f : f$ is a real valued continuous function defined on [0, 1] and $f\left(\frac{2}{3}\right) = 2\}$. Show that V is not a vector space over R under addition and scalar multiplication defined as follows :**

$$(f + g)(x) = f(x) + g(x) \text{ for all } f, g \in V$$

$$(af)(x) = af(x) \text{ for all } f \in V, a \in R.$$

Solution. Here $V = \{f : f$ is real valued continuous function defined on [0, 1] such that $f\left(\frac{2}{3}\right) = 2\}$

Let $f, g \in V$

$\Rightarrow f$ and g are real valued continuous functions defined on [0, 1] such that $f\left(\frac{2}{3}\right) = 2$ and $g\left(\frac{2}{3}\right) = 2$.

Now, $(f + g)\left(\frac{2}{3}\right) = f\left(\frac{2}{3}\right) + g\left(\frac{2}{3}\right) = 2 + 2 = 4$

$\therefore$ $f, g \in V$ does not imply, $f + g \in V$ $\left[\because (f + g)\left(\frac{2}{3}\right) \neq 2\right]$

For example,

Let $f(x) = 3x, x \in [0, 1]$

$\Rightarrow f$ is a real valued continuous functions defined on [0, 1] and

$$f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right) = 2 \Rightarrow f \in V$$

Let $g(x) = 2, x \in [0, 1]$

$\Rightarrow g$ is a real valued continuous functions defined on [0, 1] and $g\left(\frac{2}{3}\right) = 2$,

$\Rightarrow g \in V$

Now, $f + g$ is also a real valued continuous function on [0, 1]

[$\because$ *sum of two real valued continuous functions is again a real valued continuous function*]

But $$(f+g)\left(\frac{2}{3}\right) = f\left(\frac{2}{3}\right) + g\left(\frac{2}{3}\right) = 2 + 2 = 4. \Rightarrow (f+g)\left(\frac{2}{3}\right) \neq 2$$

$\therefore$ $f, g \in V$ does not imply $f + g \in V$.

Hence, V is not a vector space over R.

***Example 4.* Which of the following sets form vector spaces over reals ? If not, why ?**

(*i*) All *n*-rowed symmetric (skew-symmetric) matrices over C.

(*ii*) All upper (lower) triangular matrices of order *n* over R.

Solution. Let V(C) be the vector space of symmetric matrices over C.

Let A, B $\in$ V (C)

then $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$

Also $a_{ij} \in C, b_{ij} \in C$ and $a_{ij} = a_{ji}, b_{ij} = b_{ji}$

Now $$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]_{n \times n}$$

where $a_{ij} + b_{ij} \in C$ and $a_{ij} + b_{ij} = a_{ji} + b_{ji}$

$\therefore$ A + B is a symmetric matrix over C.

Consider $\alpha A = [\alpha a_{ij}]_{n \times n}$ for $\alpha \in C$

where $\alpha a_{ij} \in C$ and $\alpha a_{ij} = \alpha a_{ji}$

$\therefore$ αA is a symmetric matrix over C.

$\therefore$ V is closed under addition and scalar multiplication.

All other properties for a vector space are obviously satisfied. [*Verify*]

Hence, V is a vector space over C.

Similarly is the case of skew symmetric matrices over C.

(*ii*) Let V(R) be the vector space of upper triangular matrices.

Let A, B $\in$ V(R)

$\therefore$ $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$

Then $a_{ij}, b_{ij} \in R$ and $a_{ij} = 0, b_{ij} = 0$ for $i > j$

Now $A + B = [a_{ij} + b_{ij}]_{n \times n}$

Then $a_{ij} + b_{ij} = 0 + 0 = 0$ for $i > j$

and $a_{ij}, b_{ij} \in R \Rightarrow a_{ij} + b_{ij} \in R$

Hence A + B is a upper triangular matrix over R.

Consider $\alpha A = [\alpha a_{ij}]_{n \times n}$ for $\alpha \in R$

where $\alpha a_{ij} \in R$

and $\alpha a_{ij} = 0\, a_{ij} = 0$ for $i > j$

$\therefore$ αA is a upper triangular matrix over R.

Thus V is closed under addition and scalar multiplication.

All other properties for a vector space are obviously satisfied. [*Verify*]

Hence V is a vector space over R.

Similarly is the case for lower triangular matrices over R.

***Example 5.* Show that W = {(*a*, *b*, *c*) : *a*, *b*, *c* $\in$ Q} is not a subspace of V_3 (R).**

Solution. Let $W = \{(a, b, c) : a, b, c \in Q\}$

Consider $\alpha = \sqrt{2} \in R$ such that

$$\alpha(a, b, c) = (\alpha a, \alpha b, \alpha c) = (\sqrt{2}a, \sqrt{2}b, \sqrt{2}c) \notin W$$

[$\because$ $\sqrt{2}a, \sqrt{2}b, \sqrt{2}c$ *are all irrational numbers*]

$\therefore$ W is not closed w.r.t. scalar multiplication.

Hence, W is not a subspace of V_3(R).

***Example 6.* Illustrate with the help of examples that if V is a set of all 2 × 2 matrices over R, then**

(*i*) the set of all 2 × 2 singular matrices

(*ii*) the set of all matrices A for which $A^2 = A$ are not subspaces of V.

Solution. (*i*) Let W be the set of all 2 × 2 singular matrices.

Now, $$A = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \in W \quad [\because |A| = 0, |B| = 0]$$

But, $$A + B = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \notin W$$

$\therefore$ W is not a subspace of V.

(*ii*) Let W be the set of all matrices A for which $A^2 = A$.

Then, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in W$ $\quad [\because I^2 = I]$

But, $I + I = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \notin W$ $\quad \left[\because \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \neq \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right]$

$\therefore$ W is not a subspace of V.

Example 7. **Let V be a vector space over R^3. Examine whether the following are sub-spaces of V or not :**

(*i*) $W = \{(x, y, z) : x \in Z\}$

(*ii*) $W = \{(x, y, z) : x \leq 0\}$

(*iii*) $W = \{(x, y, z) : x \geq y \geq z\}$

(*iv*) $W = \{(x, y, z) : x = y = z\}$

(*v*) $W = \{(x, y, z) : y + 4z = 0\}$

(*vi*) $W = \{(x, y, z) : x = y - z \text{ and } 2x + 3y - z = 0\}$

Solution. (*i*) Let $(x, y, z) \in W$ where $x \in Z$

Let $\alpha = \sqrt{3} \in R$, then

$$\alpha(x, y, z) = (\alpha x, \alpha y, \alpha z)$$
$$= (\sqrt{3}x, \sqrt{3}y, \sqrt{3}z) \notin W \quad [\because x \in Z \text{ but } \sqrt{3}x \notin Z]$$

$\therefore$ W is not closed w.r.t scalar multiplication.

Hence, W is not a subspace of R^3.

(*ii*) Let $(x, y, z) \in W$ where $x \leq 0$

Let $\alpha = -6 \in R$, then

$$\alpha(x, y, z) = (\alpha x, \alpha y, \alpha z)$$
$$= (-6x, -6y, -6z) \notin W \quad [\because x \leq 0 \Rightarrow -6x \geq 0]$$

$\therefore$ W is not closed w.r.t scalar multiplication.

Hence, W is not a subspace of R^3.

(*iii*) Let $(x, y, z) \in W$, where $x \geq y \geq z$

Let $\alpha = -1 \in R$ then $\alpha(x, y, z) = (\alpha x, \alpha y, \alpha z)$

$$= (-x, -y, -z) \notin W \; [\because x \geq y \geq z \Rightarrow -x \leq -y \leq -z]$$

$\therefore$ W is not closed w.r.t. scalar multiplication.

Hence, W is not a subspace of R^3.

(*iv*) Let $(x, y, z) \in W$ where $x = y = z$.

Let $u = (a_1, a_1, a_1)$ and $v = (b_1, b_1, b_1) \in W$

$\therefore$ $u - v = (a_1 - b_1, a_1 - b_1, a_1 - b_1) \in W$

$$[\because a_1, b_1 \in R \Rightarrow a_1 - b_1 \in R]$$

Now, let $u \in W$ and $\alpha \in R$, then

$$\alpha u = \alpha(a_1, a_1, a_1)$$
$$= (\alpha a_1, \alpha a_1, \alpha a_1) \in W \quad [\because \alpha, a_1 \in R \Rightarrow \alpha a_1 \in R]$$

Hence, W is a subspace of R^3.

(*v*) Let $(x, y, z) \in W$ such that $y + 4z = 0$

Let $u, v \in W$ then, $u = (a_1, b_1, c_1)$ and $v = (a_2, b_2, c_2)$ such that

$$b_1 + 4c_1 = 0 \text{ and } b_2 + 4c_2 = 0$$

Now, $u - v = (a_1, b_1, c_1) - (a_2, b_2, c_2)$ such that

$$(b_1 - b_2) + 4(c_1 - c_2) = (b_1 + 4c_1) - (b_2 + 4c_2)$$
$$= 0 - 0 = 0$$

$\therefore$ $u - v \in W$

Also, $\alpha u = \alpha(a_1, b_1, c_1) = (\alpha a_1, \alpha b_1, \alpha c_1)$

such that $\alpha b_1 + 4\alpha c_1 = \alpha(b_1 + 4c_1) = \alpha(0) = 0$

$\therefore$ $\alpha u \in W$

Hence W is a subspace of R^3.

(*vi*) Let $(x, y, z) \in W$ such that $x = y - z$ and $2x + 3y - z = 0$

Let $\alpha \in R$ and $u, v \in W$

then $u = (a_1, b_1, c_1)$ and $v = (a_2, b_2, c_2)$

such that $a_1 = b_1 - c_1,\; 2a_1 + 3b_1 - c_1 = 0$ ...(1)

and $a_2 = b_2 - c_2,\; 2a_2 + 3b_2 - c_2 = 0$ ...(2)

Now $u - v = (a_1, b_1, c_1) - (a_2, b_2, c_2)$

such that $a_1 - a_2 = (b_1 - c_1) - (b_2 - c_2)$ [*Using (1) and (2)*]

$$= (b_1 - b_2) - (c_1 - c_2)$$

and $2(a_1 - a_2) + 3(b_1 - b_2) - (c_1 - c_2)$

$$= (2a_1 + 3b_1 - c_1) - (2a_2 + 3b_2 - c_2)$$
$$= 0 - 0 = 0 \quad [\textit{Using (1) and (2)}]$$

$\therefore$ $u - v \in W$

Also, $\alpha u = \alpha(a_1, b_1, c_1)$

$$= (\alpha a_1, \alpha b_1, \alpha c_1)$$

such that $\alpha a_1 = \alpha(b_1 - c_1)$

$$= \alpha b_1 - \alpha c_1$$

and $2(\alpha a_1) + 3(\alpha b_1) - \alpha c_1 = \alpha(2a_1 + 3b_1 - c_1)$

$$= \alpha(0) = 0$$

$\therefore$ $\alpha u \in W$

Hence, W is a subspace of R^3.

Example 8. **Examine whether the following are subspaces of $R^4 = \{(x, y, z, w) : x, y, z, w \in R\}$ or not :**

(*i*) $W = \{(m, m, m, m) : m \in R\}$

(*ii*) $W = \{(m, n, m, n) : m, n \in Z\}$

Solution. *(i)* Given $W = \{(m, m, m, m) : m \in R\}$

Clearly $(2, 2, 2, 2) \in W$ (as $2 \in R$)

$\Rightarrow$ W is non-empty subset of R^4

Let $u = \{(q, q, q, q) \in W : q \in R\}$

and $v = \{(r, r, r, r) \in W : r \in R\}$

For $a \in R$, we have

$$\begin{aligned} au + v &= a(q, q, q, q) + (r, r, r, r) \\ &= (aq, aq, aq, aq) + (r, r, r, r) \\ &= (aq + r, aq + r, aq + r, aq + r) \in W \end{aligned}$$

$[\because a, q, r \in R \Rightarrow aq + r \in R]$

Hence W is a subspace of R^4.

(ii) Let $(m, n, m, n) \in W$ where $m, n \in Z$

Take $a = \frac{1}{3} \in R$, $m = 3$ and $n = 2 \in Z$

Then $a(m, n, m, n) = \frac{1}{3}(3, 2, 3, 2) = \left(\frac{3}{3}, \frac{2}{3}, \frac{3}{3}, \frac{2}{3}\right) \notin W$ $\left[\because \frac{2}{3} \notin Z\right]$

$\therefore$ W is not closed under scalar multiplication.

Hence W is not a subspace of R^4

Example 9. **Examine whether the following set of vectors $u = (x_1, x_2, ..., x_n) \in R^n$ are subspaces of $R^n (n \geq 3)$?**

(i) **All u such that $x_1 \leq 0$**

(ii) **All u such that $x_1 x_2 = 0$**

(iii) **All u such that $x_3 \in Q$**

(iv) **All u such that $x_2 = x_1^2$**

(v) **All u such that $x_1 + 2x_2 = 3x_3$**

(vi) **All u such that $x_1 + x_2 + ... + x_n = \lambda$, a fixed real.**

Solution. *(i)* Let $W = \{u = (x_1, x_2, ..., x_n) \in R^n$ and $x_1 \leq 0\}$

If $\alpha \in R$ then, $\alpha u = \alpha(x_1, x_2, ..., x_n)$

Choose $\alpha = -3$ such that

$$\begin{aligned} \alpha u &= -3(x_1, x_2, ..., x_n) \\ &= (-3x_1, -3x_2, -3x_3, ..., -3x_n) \notin W \end{aligned}$$

$[\because x_1 \leq 0 \Rightarrow -3x_1 \geq 0]$

$\Rightarrow$ W is not closed w.r.t. scalar multiplication.

Hence W is not a subspace of R^n.

(ii) Let $W = \{u = (x_1, x_2, ..., x_n) \in R^n$ and $x_1 x_2 = 0\}$

Let $v = (1, 0, 0, ..., 0) \in R^n$ and $(1)(0) = 0$

$w = (0, 1, 0, ..., 0) \in R^n$ and $(0)(1) = 0$.

Now $v + w = (1, 0, 0, ..., 0) + (0, 1, 0, ..., 0)$

$= (1, 1, 0, ..., 0) \notin W$ $[\because (1)(1) = 1 \neq 0]$

$\Rightarrow$ W is not closed w.r.t. addition.

Hence W is not a subspace of R^n.

(iii) Let $W = \{u = (x_1, x_2, x_3, ..., x_n) \in R^n$ and $x_3 \in Q\}$

Let $u = (x_1, x_2, x_3, ..., x_n)$ and $v = (y_1, y_2, y_3, ..., y_n) \in R^n$

where $x_3, y_3 \in Q$.

Now for $a \in R$, we have

$$\begin{aligned} au + v &= a(x_1, x_2, x_3, ..., x_n) + (y_1, y_2, y_3, ..., y_n) \\ &= (ax_1 + y_1, ax_2 + y_2, ax_3 + y_3, ..., ax_n + y_n) \notin W \end{aligned}$$

$[\because$ *If* $a = \sqrt{3}, x_3 = 2, y_3 = 4$ *then* $ax_3 + y_3 = 2\sqrt{3} + 4 \notin Q]$

$\therefore$ W is not a subspace of R^n.

(iv) Let $W = \{u = (x_1, x_2, ..., x_n) \in R^n$ and $x_2 = x_1^2\}$

Let $X = (2, 4, 0, ..., 0)$ and $Y = (3, 9, 0, ..., 0) \in W$.

Now $x_2 = 4 = 2^2 = x_1^2$

and $y_2 = 9 = 3^2 = y_1^2$

Now $X + Y = (2, 4, 0, ..., 0) + (3, 9, 0, ..., 0)$

$= (5, 13, 0, ..., 0) \notin W$ $[\because 13 \neq (5)^2]$

$\Rightarrow$ W is not closed w.r.t. addition.

Hence W is not a subspace of R^n.

(v) Let $W = \{u = (x_1, x_2, ..., x_n) \in R^n$ and $x_1 + 2x_2 = 3x_3\}$

Let $u = (x_1, x_2, ..., x_n)$ and $v = (y_1, y_2, ..., y_n) \in W$

where $x_1 + 2x_2 = 3x_3$ and $y_1 + 2y_2 = 3y_3$

Let $a \in R$, then

$$\begin{aligned} au + v &= a(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n). \\ &= (ax_1, ax_2, ax_3, ..., ax_n) + (y_1, y_2, ..., y_n). \\ &= (ax_1 + y_1, ax_2 + y_2, ax_3 + y_3, ..., ax_n + y_n) \end{aligned}$$

where
$$\begin{aligned} (ax_1 + y_1) + 2(ax_2 + y_2) &= (ax_1 + 2ax_2) + (y_1 + 2y_2) \\ &= a(x_1 + 2x_2) + (y_1 + 2y_2) \\ &= 3ax_3 + 3y_3 = 3(ax_3 + y_3) \end{aligned}$$

$\therefore \quad au + v \in W$ for $a \in R$ and $u, v \in W$.

Hence W is a subspace of R^n.

(*vi*) Let $\quad W = \{u = (x_1, x_2, \ldots\ldots, x_n) \in R^n$ and $x_1 + x_2 + \ldots\ldots + x_n = \lambda$, a fixed real$\}$.

Case I. When $\lambda \neq 0$

Let $\quad v = (3, 4, 0, \ldots\ldots, 0)$

where $\quad v_1 + v_2 + \ldots\ldots + v_n = 7 \neq 0$

and $\quad w = (-6, -1, 0, \ldots\ldots, 0)$

where $\quad w_1 + w_2 + \ldots\ldots + w_n = -7 \neq 0$

then $\quad v + w = (3, 4, 0, \ldots\ldots, 0) + (-6, -1, 0, \ldots\ldots, 0)$

$= (-3, 3, 0, \ldots\ldots, 0)$ where $-3 + 3 + 0 + \ldots\ldots + 0 = 0$

$\in W$

$\therefore \quad$ W is not a subspace of R^n, for $\lambda \neq 0$.

Case II. When $\lambda = 0$

Let $\quad u = (x_1, x_2, \ldots\ldots, x_n)$ and $v = (y_1, y_2, \ldots\ldots, y_n) \in W$

such that $x_1 + x_2 + \ldots\ldots + x_n = 0$ and $y_1 + y_2 + \ldots\ldots + y_n = 0$

Let $a \in R$

Then $\quad au + v = (ax_1 + y_1, ax_2 + y_2, \ldots\ldots, ax_n + y_n)$

such that $(ax_1 + y_1) + (ax_2 + y_2) + \ldots\ldots + (ax_n + y_n)$

$$= a\,(x_1 + x_2 + \ldots\ldots + x_n) + (y_1 + y_2 + \ldots\ldots + y_n)$$

$$= a(0) + 0 = 0$$

$\Rightarrow \quad au + v \in W$ for $a \in R$ and $u, v \in W$.

$\therefore \quad$ W is a subspace of R^n, for $\lambda = 0$.

Example 10. **Let $V = \{f(x) : f(x)$ is a polynomial of degree $\leq 6\}$. Illustrate with the help of examples that which of the following are subspaces :**

(*i*) $\quad$ **$W = \{f(x) : \deg f(x) = 5\}$**

(*ii*) $\quad$ **$W = \{f(x) : \deg f(x) \leq 3\}$**

(*iii*) $\quad$ **$W = \{f(x) : f(0) = 4\}$**

(*iv*) $\quad$ **$W = \{f(x) : f(2) = 0 = f(5)\}$**

(*v*) $\quad$ **$W = \{f(x)$: Coefficient of x^2 is 5 or $-5\}$**

(*vi*) $\quad$ **$W = \{f(x) : f(x)$ has $-$ ve co-efficients$\}$**

Solution. (*i*) Let $W = \{f(x) : \text{degree } f(x) = 5\}$.

Let $\quad f(x) = x^5 + 2x^4 - x^3 - 1 \in W$

and $\quad g(x) = -x^5 + x^4 - x^3 + 2x^2 + 5 \in W$

But $\quad f(x) + g(x) = 3x^4 - 2x^3 + 2x^2 + 4 \notin W$

[$\because$ *It is not a polynomial of degree 5*]

Hence W is not a subspace of V.

(*ii*) Let $\quad W = \{f(x) : \text{degree } f(x) \leq 3\}$

Let $\quad f(x) = x^2 + x + 1 \in W$

and $\quad g(x) = 3x^2 - 3x + 5 \in W$

Let $a \in R$ then $a\,f(x) + g(x)$ is a polynomial of degree ≤ 3.

$\therefore \quad af(x) + g(x) \in W$

[$\because$ *$f(x)$ and $g(x)$ both are polynomials of degree ≤ 3*]

Hence W is a subspace of V.

(*iii*) Let $\quad W = \{f(x) : f(0) = 5\}$.

Since zero polynomial does not belong to W as for zero polynomial

$$f(0) = 0 \quad \text{and} \quad f(0) \neq 5.$$

W is not a subspace of V.

(*iv*) Let $\quad W = \{f(x) : f(2) = 0 = f(5)\}$

Let $\quad f(x) = 2x^2 - 14x + 20$,

which is a polynomial of degree 2 and $f(2) = 0 = f(5)$.

$\therefore \quad f(x) \in W$.

Now let $f(x), g(x) \in W$ and $a \in R$

where $\quad f(2) = 0 = f(5)$

and $\quad g(2) = 0 = g(5)$

Now, let $a\,f(x) + g(x)$ is a polynomial in x and is denoted by $F(x)$

such that $\quad F(2) = af(2) + g(2) = a(0) + 0 = 0$.

and $\quad F(5) = af(5) + g(5) = a(0) + 0 = 0$

Hence $\quad F(x) \in W$

$\Rightarrow \quad af(x) + g(x) \in W$ for all $a \in R$ and $f(x), g(x) \in W$.

Hence, W is a subspace of V.

(*v*) Let $W = \{f(x)$: coefficient of x^2 is 5 or $-5\}$.

Since zero polynomial does not belong to W.

Since coefficient of x^2 in zero polynomial is 0.

Hence, W is not a subspace of V.

(*vi*) Let $W = \{f(x) : f(x)$ has $-$ve co-efficients$\}$.

Since zero polynomial does not belong to W as all coefficients are zero in zero polynomial.

$\therefore$ W is not a subspace of V.

Example 11. **Let V be a vector space of all real valued continuous functions over R. Show that the set W of solutions of differential equations $3\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2y = 0$ is a subspace of V.**

Solution. Let $W = 3\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2y = 0$.

Since $W \neq \phi$ and $y = 0$ satisfies the above differential equation.

We have to show that W is a subspace of V.

Let $y_1, y_2 \in W$ and $a \in R$, such that

$$3\frac{d^2y_1}{dx^2} - 5\frac{dy_1}{dx} + 2y_1 = 0 \quad ...(1)$$

and

$$3\frac{d^2y_2}{dx^2} - 5\frac{dy_2}{dx} + 2y_2 = 0 \quad ...(2)$$

Multiplying equation (1) by a, we get

$$3\frac{d^2y_1a}{dx^2} - 5\frac{dy_1a}{dx} + 2y_1a = 0 \quad ...(3)$$

Adding (2) and (3), we get

$$3\frac{d^2}{dx^2}(ay_1 + y_2) - 5\frac{d}{dx}(ay_1 + y_2) + 2(ay_1 + y_2) = 0$$

$\therefore$ $ay_1 + y_2$ is a solution of the given differential equation.

$\therefore$ $ay_1 + y_2 \in W$

Hence, W is a subspace of V.

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