Roll No. ...20311230006 4   Exam Code : M-23

Subject Code—59296

B.Sc. EXAMINATION

(Main/Reappear) (Batch 2018 Onwards)

(Sixth Semester)

MATHEMATICS

CML-607(i)

Real and Complex Analysis

Time : 3 *Hours*   *Maximum Marks* : 80

Note : Attempt *Five* questions in all, selecting *one* question from each Section. Q. No. 1 is compulsory. All questions carry equal marks.

1. (a) Explain Semi-Metric Space with example.

   (b) If A and B are subsets of a metric space (X, d), then $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$.

   (c) Define Isometry. Also prove that Isometry is a uniformly continuous function.

(d) Prove that :

$$\int_0^{\infty} \frac{\tan^{-1} ax - \tan^{-1} bx}{x} dx = \frac{\pi}{2} \log\left|\frac{a}{b}\right|.$$

(e) Discuss the convergence of the improper integral $\int_a^{\infty} \frac{dx}{x^n} (a > 0)$.

(f) Check the derivability of the function $f(z) = \overline{z}$ in complex plane.

(g) If $w = f(z)$ is a regular function of z, prove that :

$$\left(\frac{\partial^2}{\partial^2 x^2} + \frac{\partial^2}{\partial^2 y^2}\right) \log|f'(z)| = 0.$$

Section I

2. (a) Let (X, d) be a metric space and let (Y, d^*) be a subspace thereof. A subset B of Y is d^*-open iff there exists a d^*-open subset G of X such that B = G $\cup$ Y.

(b) Define Metric space. Prove that the set of all continuous real valued functions defined on the closed and bounded interval $[a, b]$ has a metric defined by $d(f,g) = \sup\{|f(x) - g(x)| : x \in [a, b]\}$.

3. (a) Show that the interior set of subset of a metric space is the largest open set contained in A.

(b) Prove that the metric space (R, d) is complete, where d denotes the usual metric.

Section II

4. (a) If (X, d) and (Y, d^*) be metric spaces and Y be complete. Let A be dense subset of X. If $f : A \to Y$ be uniformly continuous on A, then f can be extended uniquely to a uniformly continuous function $g : X \to Y$.

(b) Prove that a metric space is sequentially compact iff it has the Bolzano-Weierstrass property.

5. (a) A metric space (X, d) is compact iff every collection of closed subsets of X with finite intersection property has a non-empty intersection.

(b) A metric space (X, d) is disconnected iff there exists a non empty proper subset of X which is both d-open and d-closed.

Section III

6. (a) Discuss the convergence of the integral :

$$\int_0^\infty x^{n-1} e^{-x} dx$$

(b) Examine the convergence of the integral :

$$\int_0^\infty \frac{\sin x}{x^p} dx.$$

7. (a) Evaluate the integral :

$$\int_0^\infty \frac{\tan^{-1} \alpha x}{x(1+x^2)} dx, \text{ where } \alpha \geq 0.$$

(b) Show that :

$$\int_0^\infty \frac{\sin ax \sin bx}{x} dx = \frac{1}{2} \log \frac{a+b}{a-b},$$

where $a > b > 0$.

Section IV

8. (a) Show that the function :

$$f(z) = \begin{cases} \dfrac{(\overline{z})^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is not differentiable at origin, although the C-R equations are satisfied at that point.

(b) Show that $u = \frac{1}{2} \log(x^2 + y^2)$ is harmonic and find its harmonic conjugate.

9. (a) If $f(z)$ is an analytic function of z, find $f(z)$ if $u - v = (x - y)(x^2 + 4xy + y^2)$.

(b) Prove that an analytic function with constant modulus is constant.