Compulsory Question

1. (a) Define Semi-Metric space (Pseudo-Metric space). 2

   (b) Prove that the usual metric space (R, *d*) is not compact. 2

   (c) Define Improper Integral and types of improper integral with examples. 2

   (d) Show that $f(z) = z^2$ is uniformly continuous in the region $|z| < 1$. 3

   (e) Define Analytic function and Harmonic function with examples. 2

   (f) Examine the convergence of the improper integral : 3

   $$\int_0^{\infty} \frac{dx}{1+x^2}$$

   (g) Let (X, *d*) be a discrete metric space. Then prove that the only connected sets are ϕ and singleton set. 2

Unit I

2. (a) Let (X, *d*) be a metric space. Define $d^* : X \times X \to R$ by $d^*(x, y) = \min\{2, d(x, y)\}$. 8

   (b) Every derived set is a closed set. 8

3. (a) Every convergent sequence in a metric space is a Cauchy sequence but converse need not be true. 8

   (b) Prove that the usual metric space (R, *d*) is complete. 8

Unit II

4. (a) If $\{A_n\}$ is a sequence of nowhere dense sets in a complete metric space X, then there exists a point in X, which is not in any of the A_n's. 8

   (b) Let (X, *d*) and (Y, d^*) be two metric spaces and let $f : X \to R$. Then f is continuous iff the inverse image under 'f' of every closed subset of Y is a closed subset of X. 8

5. (a) Prove that the infinite discrete metric space (X, d) is neither sequentially compact nor possesses the BWP (Bolzano Weierstrass Property). 8

(b) If E is a connected subset of a metric space (X, d) such that $E \subset A \cup B$, where A and B are separated sets in X, then either $E \subset A$ or $E \subset B$. Prove. 8

Unit III

6. (a) Discuss the convergence of the Beta function. 8

(b) Discuss the convergence of the integral

$$\int_0^1 \frac{dx}{x^3(1+x^2)^5}.$$ 8

7. (a) State and prove Dirichlet's Test for Convergence. 8

(b) Show that : 8

$$\int_0^1 \frac{x^{\alpha}-1}{\log x}dx = \log(1+\alpha)$$

Unit IV

8. (a) Prove that an analytic function with constant modulus is constant. 8

(b) Find the regular function whose imaginary part is $v = \frac{x-y}{x^2+y^2}$. 8

9. (a) Show that the function $f(z) = |z|^2$ is continuous everywhere but nowhere differentiable except at origin. 8

(b) Separate $\log \sin(x + iy)$ into real and imaginary parts. 8