(b) Define minimal polynomial. Let $x^n - a \in F[x]$ be an irreducible polynomial over F, and let $b \in K$ be its root, where K is an extension of F. If m is a positive integer such that $m|n$. Find the degree of the minimal polynomial of b^m over F.

7. (a) If $f(x) \in F[x]$ is irreducible polynomial over F, then prove that all roots of $f(x)$ have the same multiplicity.

(b) Construct finite fields of order 8 and 9.

Unit IV

8. Let H be a finite subgroup of the group of automorphisms of a field E. Then prove that $[E : E_H] = |H|$, where symbols have their usual meanings.

9. (a) Prove that the Galois group of $x^4 - 2 \in Q[x]$ is the octic group.

(b) Let E be a splitting field of $f(x) \in F[x]$ such that G(E/F) is solvable. Then prove that $f(x)$ is solvable by radicals.

Roll No. .... **Exam Code : J-21**

Subject Code—5541

M. Sc. EXAMINATION

(Batch 2017 Onwards) (Regular)/(Batch 2019 Onwards) (Distance)

(First Semester)

MATHEMATICS

MAL-511

Algebra

Time : 3 Hours *Maximum Marks :* 70

Note : Attempt *Five* questions in all. Q. No. **1** is compulsory. All questions carry equal marks.

1. (a) Write a composition series of the quaternion group.

(b) Whether S_3 is solvable. If yes, justify your answer.

(c) Define Prime field. Give *one* example each of finite and infinite prime field, if any.

(d) What do you mean by transcendental extension ? Give *one* example.

(e) Define algebraic closure. What is the algebraic closure of real number field R ?

(f) Define Cyclotonic polynomial. What is $\Phi_4(x)$?

(g) Prove that if 'a' is a constructible number, then x = 'a' and y = 'a' are constructible lines.

Unit I

2. State and prove Zassenhaus's lemma and using this prove Schreier's theorem.

3. (a) If G is a cyclic group such that $|G| = p_1, p_2, \ldots\ldots p_r$, p_i being distinct primes. Then show that the number of distinct composition series of G is r_0'.

(b) Let G be a group and $a, b, c \in G$. Denote $a^b = b^{-1}ab$. Then prove that :

$$\left[a, b^{-1}, c\right]^b \left[b, c^{-1}, a\right]^c \left[c, a^{-1}, b\right]^a = 1$$

Unit II

4. (a) Define simple and solvable groups. Show that a simple group is solvable if and only if it is cyclic.

(b) Show that the index of a maximal subgroup H of a solvable group G is either infinite or a prime power.

5. (a) Let G be a nilpotent group. Then show that every subgroup of G and every homomorphic image of G are nilpotent.

(b) Prove that the prime field of a field F is either isomorphic to Q or Z/(p), where p is a prime.

Unit III

6. (a) Find a suitable number 'a' such that $Q\left(\sqrt{5}, i\right) = Q(a)$.