5. (a) Prove that :

$$\cosh\left(z+\frac{1}{z}\right)=a_0+\sum_{n=1}^{\infty}a_n\left(z^n+\frac{1}{z^n}\right),$$

where $a_n=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\cosh(2\cos\theta)\cos n\theta d\theta.$

(b) Expand $f(z)=\dfrac{1}{1+z}$ in a Taylor series centered at $z_0 = -i$. Find the radius of the resulting series.

Unit III

6. (a) Show that an analytic function comes arbitrary close to any complex value in the neighbourhood of an essential singularity.

(b) Use Rouche theorem to show that all the seven zeros of $z^7 + 10z^3 + 14$ lie within the annulus $1 < |z| < 2$.

7. (a) State and prove Cauchy's residue theorem.

Subject Code—5545

M. Sc. EXAMINATION

(Batch 2017 Onwards)

(First Semester)

MATHEMATICS

MAL-515

Complex Analysis-I

Time : 3 *Hours* *Maximum Marks :* 70

Note : Attempt any *Five* questions. Q. No. **1** is compulsory. All questions carry equal marks.

1. (a) What do you mean by an analytic function ?

(b) Find the kind of singularity of $\sin\dfrac{1}{1-z}$ at $z = 1$.

(c) Find the residue of $\frac{z^3}{z^2-1}$ at $z = \infty$.

(d) What is meant by conformal mapping ?

(e) What do you mean by multiply connected region ?

(f) Show that the function $f(z) = x + 4iy$ is not differentiable at any point z.

(g) Find the fixed points of the bilinear transformation $w = \frac{(2+i)z-2}{z+i}$.

Unit I

2. (a) State and prove the necessary condition for $f(z)$ to be analytic.

(b) Examine the nature of the function :

$$f(z) = \begin{cases} x^2y^5(x+iy)/(x^4+y^{10}), & z \neq 0 \\ 0 & , \quad z = 0 \end{cases}$$

in the region including the origin.

3. (a) State Cauchy integral formula and hence evaluate :

$$\int_C \frac{e^z}{(z^2+\pi^2)^2} dz,$$

where C is $|z| = 4$.

(b) Let $f(z)$ be analytic within and on the boundary C of a simply connected region D and let z_0 be any point within C. Show that the derivatives of all orders of $f(z)$ are analytic and given by :

$$f^{(n)}(z_0) = \frac{\underline{|n}}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}}$$

Unit II

4. (a) Define complex integral as a function of its upper limit and hence state and prove Morera's theorem.

(b) State and prove maximum modulus principle.

(b) Using calculus of residue, evaluate :

$$\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\, d\theta$$

Unit IV

8. (a) Show that $\log z$ is a many-valued function and has infinitely many values.

(b) Describe an expression which can be used to transform one given circle into another given circle or straight line.

9. (a) Find the images of the points 0, 1 + i, i and ∞ under the bilinear transformation $T(z) = \frac{2z+1}{z-i}$. Also explain, why bilinear transformation is named so ?

(b) State and prove the necessary condition for $w = f(z)$ to represent a conformal mapping.

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