Subject Code—18001

M.Sc. EXAMINATION

(Batch 2020 Onwards)

(Second Semester)

MATHEMATICS

MAL-521

Abstract Algebra

Time : 3 *Hours* *Maximum Marks* : 70

Note : Attempt *Five* questions in all, selecting *one* question from each Section. Q. No. **1** is compulsory.

(Compulsory Question)

7. (i) Define an invariant subspace of V, invariant under $S \in A(V)$. Also give an example of it. **2**

(ii) Write two different nilpotent transformations with index of nilpotence 3 acting on a five dimensional vector space. 2

(iii) Write the minimal polynomial of the transformation $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ over the field of rationals. 2

(iv) Define free module. Is every cyclic module free ? 2

(v) Give an example of a ring which is artinian but not Noetherian. 2

(vi) Define primary module with an example. 2

(vii) Let G be a finitely generated abelian group generated by x and y subject to the conditions $2x = 0$ and $3y = 0$. Find n so that $G \cong Z_n$. 2

Section I

2. (a) Show that every nilpotent transformation T is singular while T + 5I is non-singular. 5

(b) If subspace W of vector space is invariant under T, then T induces a linear transformation T* on $\frac{V}{W}$, defined by $(v + W)T^* = vT + W$. Further if T satisfies $q(x)$ over F, then so does T*. 9

3. (a) If $T \in A(V)$ is a nilpotent transformation with index of nilpotence m, then there exists invariant subspaces U and W of V having dimensions m and dim V-m respectively, such that $V = U \oplus W$. Also show that U is invariant under T. 8

(b) Show that the invariants of a nilpotent transformation are unique. 6

Section II

4. (a) If $p(x) = p_1(x)^{t_1}, p_2(x)^{t_2} \ldots p_k(x)^{t_k}$, where $p_i(x)$ are irreducible over F, is the minimal polynomial of $T \in A(V)$, then show that V is direct sum of k different cyclic subspaces. 7

(b) If all the distinct characteristic roots $\lambda_1, \ldots, \lambda_k$ of $T \in A(V)$ lies in F, then a basis of V can be found in which the matrix of T is of the form

$$\begin{bmatrix} J_1 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & J_k \end{bmatrix}$$ where each

$$J_i = \begin{bmatrix} B_{i1} & 0 & 0 & 0 \\ 0 & B_{i2} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & B_{ir_i} \end{bmatrix}$$ and where

$B_{i1}, B_{i2}, \ldots, B_{ir_i}$ are basic Jordan block belonging to λ. 7

5. (a) Let $p(x)$ be the minimal polynomial of $T \in A(V)$ over F. If V is cyclic F[x]-module, then there exist a basis of V in which the matrix of T is the companion matrix of $f(x)$. 8

(b) Define the elementary divisors of a transformation $T \in A(V)$. Write the rational canonical form of the 6×6 order matrix whose elementary divisors are $(x-4)$, $(x-4)(x-3)$, $(x-4)(x-3)^2$. 6

Section III

6. (a) Define Noetherian module over a ring R. Show that every submodule, factor module and homomorphic image of a Noetherian module is Noetherian. 9

(b) State and prove Schurs lemma. 5

7. (a) Let M be an artinina module over a ring R. Show that the following statements are equivalent :

   (i) M is artinian

   (ii) Every quotient module of M is finitely cogenerated

   (iii) Every collection of submodules of M has a minimal element. **9**

   (b) Show that a subring of an Artinian ring need not be Artinian. **5**

Section IV

8. (a) Let $M = \oplus \sum_{i=1}^{k} M_i$ be a direct sum of R-modules M_i. Then $\text{Hom}_R(M, M) \cong T$, where T is a ring of $K \times K$ matrices $[f_{ij}]$ where f_{ij} is an element of $\text{Hom}_R(M_j, M_i)$. **9**

   (b) Let B be a minimal left ideal of a ring R. Show that either B is nilpotent or, it is generated by an idempotent θ in R. **5**

9. (a) State and prove Wedderburn Artin theorem. **10**

   (b) Get the smith normal form for $\begin{bmatrix} 0 & 2 & -1 \\ -3 & 8 & 3 \\ -2 & 4 & 1 \end{bmatrix}$ over the ring of integers. **4**