MSc - 2ND Semester

Roll No. .... **Exam Code : N-21**

Jan. 2022

Subject Code—11200

M. Sc. EXAMINATION

(Main & Re-appear)

(For Batch 2019 Onwards)

(Second Semester)

MATHEMATICS

MAL-521

Abstract Algebra

Time : 3 Hours *Maximum Marks* : 70

Note : Attempt *Five* questions in all, selecting *one* question from each Unit. Q. No. **1** is compulsory.

1. (a) Give *two* examples of linear transformations having same canonical form. **2**

(b) Does the transformations defined by $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ have same invariants. 2

(c) What do you understand by primary decomposition theorem on a vector space ? 2

(d) Give an example of a module which is (i) Free module (ii) not a free module. 2

(e) Give *two* examples of semi-simple modules. 2

(f) Write the statement of Noether-Lasker theorem. 2

(g) Let M be a simple R-module. Find the order of $Hom_r(M, M)$. 2

Unit I

2. (a) Let W be an invariant subspace of V, invariant under T. Define a transformation S on $\frac{V}{W}$ so that S and T satisfy same minimal polynomial. 7

(b) Show that a transformation T have a unique set of invariants. 7

3. (a) Define invariants of a transformation. Show that two transformations are similar if and only if they have same invariants. 7

(b) Let V be a seven dimensional vector space over field F. Write all the possible invariants of a transformation whose matrix under a basis is $\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$.

7

Unit II

4. (a) Let $p(x), q(x)$ and $r(x)$ be irreducible over F. Further, let $t(x) = p(x)q(x)r(x)$ be the minimal polynomial of T. Using $t(x)$ write V as direct sum of some invariant subspaces of V. 7

(b) Explain Jordan canonical form of a transformation acting on a finite dimensional vector space. Let $x^3 - 18x^2 + 107x - 210$ be the minimal polynomial of T acting on a vector space of dimension 3. Then write its Jordan form. 7

5. (a) Let $x^n - \delta_1 x^{n-1} - \delta_2 x^{n-2} - \ldots - \delta_n$ be the minimal polynomial of $T \in A(V)$. Show that there exists a basis of V in which matrix of T is :

$$\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \varepsilon_1 & \delta_{n-1} & \delta_{n-2} & \ldots & \delta_1 \end{bmatrix}.$$ 7

(b) Show that elementary divisors of a transformation are unique. 7

Unit III

6. (a) Let M be a finitely generated free module over a commutative ring R. Show that all the basis of M are finite. Moreover all bases of M have same number of elements. 8

(b) Let R be a commutative ring with unity. If R has an idempotent other than zero and 1, then show that R*e* cannot be a free R–module. 6

7. (a) Show that every noetherian module is a finitely generated module. Give *two* examples of noetherian modules. Also give *two* examples of modules which are not noetherian. 7

(b) Show that a module M is artinian if and only if every quotient module of it is finitely co-generated. Give *two* examples of artinian modules. Also give *two* examples of non-aritinian modules. 7

Unit IV

8. (a) Let R be a right artinian ring with unity. If R has no non-zero ideal, then show that R is isomorphic to a finite direct sum of matrix rings over division rings. 8

(b) Let M be a minimal left ideal in a ring R. If $m\mathrm{M} \neq (0)$, then show that M = Re, where e is an element of R such that $e^2 = e$. 6

9. (a) Let R be a noetherian ring. Show that every module over R contains a uniform module. 7

(b) Obtain the Smith normal form and rank for the matrices :

$$\begin{bmatrix} 0 & 2 & -1 \\ -3 & 8 & 3 \\ 2 & -4 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} -x-3 & 2 & 0 \\ 1 & -x & 1 \\ 1 & -3 & -x-2 \end{bmatrix}$$

over Z and $\mathbb{Q}[x]$ respectively. 7