Subject Code—26115

M.Sc. EXAMINATION

(Batch 2025 Onwards Under NEP-2020)

(First Semester)

MATHAMATICS

U25MAT101T

Abstract Algebra-I

Time : 3 *Hours* *Maximum Marks* : 70

Note : Question paper contains nine questions in all. Attempt *Five* questions in all selecting *one* question from each Section and the compulsory question i.e. Q. No. **1** question.

Compulsory Question

1. (a) Write all the elements of order 5 in S_5, symmetric group of degree 5. **2**

(b) Find the third commutator of D_6, where D_6 is the dihedral group of order 12. **2**

(c) Write all the composition series of group of quaternions. 2

(d) Show that every nilpotent group is solvable. 2

(e) Give an example of nilpotent group of class 4. 2

(f) Show that the set of all rational numbers is a prime field. 2

(g) Write the algebraic closure of the field of real numbers. 2

Section I

2. (a) Let G be an external direct product of the groups G_1 G_2, G_3. Show that G can also be written as internal direct product of subgroups of G isomorphic to G_1, G_2, G_3. 7

(b) Prove that the center of a p-group is always nontrivial. Hence or otherwise prove that every group of order p^2 is abelian. 7

3. (a) If a prime p divides the order of a group G, show that G has an element of order p.. 7

(b) Prove that a group of order n is not a simple group when $n = 36, 56$ or 63. 7

Section II

4. (a) State and prove Zessenhaus Lemma. 7

(b) Prove that every finite group has a composition series. Obtain it for D_{10}. Also write all the composition series of D_4, where D_n is the Dihedral group of degree n. 7

5. (a) State and prove the Jordan Holder theorem. Also prove that the Jordan Holder theorem implies that every number $n > 1$ can be written as a product of primes in a unique way. 7

(b) Write $D_{100}^{(1)}, D_{100}^{(2)}, D_{100}^{(3)}$, where $D_{100}^{(n)}$ denotes the nth commutator subgroup of a Dihedral subgroup of degree 100. 7

Section III

6. (a) Define a nilpotent group. Show that every p group, where p is a prime number, is nilpotent. For $n > 2$, give an example of a dihedral group which is nilpotent. Also give an example of Dihedral group which is not nilpotent. 9

(b) Let G be a nilpotent group of class n. Show that the length of upper central series and lower central series of G is n. 5

7. (a) Define solvable group. Show that a group G is solvable if and only if H and G/H, where H is a normal subgroup of G, are solvable. 9

(b) Give an example of group G so that the commutator subgroup of G is G itself. Hence or otherwise prove that the G is not nilpotent. 5

Section IV

8. (a) Define a prime field. Show that the Z_p where p is a prime, is a prime field. 3

(b) State and prove finite extension theorem on fields. 7

(c) Show that : 4

$$Q(\sqrt{3}, \sqrt[3]{5}) = Q(\sqrt{3} + \sqrt[3]{5})$$

9. (a) Define Algebraic extension, Algebraically closed and the Algebraic closure of a field. Also show that every finite extension of a field is algebraic but converse may not be true. 10

(b) Find the splitting field of $x^{25} + x^5 + 1$ over the field of rational numbers. 4

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