Exam Code : M-26

Subject Code—30602

M. Sc. EXAMINATION

(Second Semester)

(Batch 2025 Onwards)

(UTD and Affiliated Colleges)

(Under NEP-2020)

MATHEMATICS

U25MAT202T

Lebesgue Measure and Integration

Time : 3 *Hours* *Maximum Marks* : 70

Note : Attempt *Five* questions in all, selecting *one* question from each Unit. Q. No. **1** is compulsory. All questions carry equal marks.

1. (a) Show that outermeasure of a null set is zero.

(b) Define characteristic function.

(c) State Littlewood's first principle.

P.T.O.

(d) Define Lebesgue Integral as integral of a simple function

(e) Define integral of non-negative measurable functions.

(f) State fundamental theorem of integral calculation.

(g) Define Dini's Derivatives.

Unit I

2. (a) Prove that Outermeasure is countably subadditive.

(b) Let A be any set and $E_1, E_2, \ldots\ldots\ldots, E_n$ be a finite sequence of disjoint measurable sets, then prove that :

$$m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

3. (a) Prove that the family M of all measurable set is a σ-algebra.

(b) Show that Contor's Ternary set has measure zero.

Unit II

4. (a) Let f and g be measurable real valued function on E then show that :

   (i) $f + g$

   (ii) $f - g$

   (iii) $|f|$

   are measurable.

   (b) Let f and g be two functions defined on a common domain E such that $f = g$ a.e. and g is measurable. Then show that f is measurable.

5. (a) Let E be a measurable set with $mE < \infty$ and $\{f_n\}$ be a sequence of measurable functions converging almost everywhere to a real valued function f defined on E. Then show that, given $\in > 0$, $\delta > 0$ there is a set $A \subset E$ with $mA < \delta$ and an integer N such that.

   $$|f_n(x) - f(x)| < \in \ \forall \ x \in E - A \text{ and } n \geq N$$

   (b) Let f be a measurable function defined on E, then prove that there exists a sequence $\{g_n\}$ of continuous functions on R such that :

   $$g_n \to f \text{ a.e. on E.}$$

Unit III

6. (a) Let f be a bounded function defined on $[a, b]$. If f is Riemann integrable over $[a, b]$, then show that it is Lebesgue integrable and

$$R\int_a^b f(x)\,dx = L\int_a^b f(x)\,dx.$$

(b) Let f be a non-negative function which is integrable over a set E. Then show that given $\in > 0$, there exists a $\delta > 0$ such that for every set $A \subset E$ with $m(A) < \delta$, we have $\int_A f < \in$.

7. (a) State and prove Fatou's Lemma.

(b) Define the Lebesgue integral for general measuable functions. Explain integrability conditions.

Unit IV

8. State and prove Lebesgue differentiation theorem.

9. (a) Prove that a function F is an indefinite integral if and only if F is absolutely continuous function.

(b) If f and g are two functions of bounded variation on $[a, b]$ and λ is constant. Then show that λf and $f + g$ are function of bounded variation on $[a, b]$.