Subject Code—18002

M.Sc. EXAMINATION

(Batch 2020 Onwards)

(Second Semester)

MATHEMATICS

MAL-522

Measure and Integration Theory

Time : 3 *Hours* *Maximum Marks* : 70

Note : Attempt *Five* questions in all, selecting *one* question from each Unit. Q. No. **1** is compulsory. All questions carry equal marks.

1. (a) Define measurable function. If f be a measurable function defined on a set E, then prove that the set $E(f = \alpha)$ is measurable for each extended real number α.

(b) Explain the measurability of function $f : [0, 1] \to \mathbf{R}$:

$$f(x)\begin{cases} 1/x & 0 < x < 1 \\ 5 & x = 0 \\ 7 & x = 1 \end{cases}.$$

(c) Prove that convergence almost everywhere (a.e.) implies convergence in measure.

(d) State monotone convergence theorem. Explain with an example that monotone convergence theorem need not hold good for decreasing sequence of functions.

(e) Define Lebesgue point. Prove that every point of continuity of an integrable function f is a Lebesgue point of f.

(f) If $\int_E f = 0$ and $f(x) \geq 0$ on E, then $f = 0$ a.e.

(g) Define Classical L^P-space. Let $E = (0, \infty)$ and the function $f : E \to R$ defined by $f(x) = \dfrac{1}{\sqrt{(1+x)}}$, then $f \in L^p(E)$ for each $p, 2 < p < \infty$.

Unit I

2. (a) Let E be a measurable set with $m(E) < \infty$. If $\{f_n\}$ be a sequence of measurable functions on E and f be a measurable function such that $f_n(x) \to f$ on E then given $\epsilon > 0$ and $\delta > 0$ there is a measurable set $A \subset E$ with $m(A) < \delta$ and an integer N such that : $|f_n(x) - f(x)| < \epsilon$ for all $x \in E - A$ and all $n \geq N$.

(b) If f is a measurable real valued function on a measurable set E and g be a function defined and continuous in the range of f then (gof) is a measurable function on E.

3. (a) If $\{f_n\}$ is a sequence of measurable functions which is fundamental in measure, than there exists a measurable function f such that $\{f_n\}$ converges in measure to f.

(b) Define convergence in measure. State and Prove F. Riesz theorem for convergence in measure.

Unit II

4. (a) A bounded function f defined on a measurable set E of finite measure is Lebesgue integrable iff f is measurable.

(b) State and prove Fatou's Lemma.

5. (a) Let f be a non-negative function which is integrable over a set E. Then given $\varepsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $m(A) < \delta$, we have $\int_A f < \varepsilon$.

(b) State and prove Lebesgue dominated convergence theorem.

Unit III

6. (a) State and prove Vitali covering theorem.

(b) A function f defined on $[a, b]$ is of bounded variations iff it can be expressed as a difference of two monotonic increasing functions on $[a, b]$.

7. (a) Let f be bounded and measurable function defined on $[a, b]$. If

$$F(x) = \int_a^x f(t)dt + F(a), \text{ then } F'(x) = f(x)$$

a.e. on $[a, b]$.

(b) If f is a absolutely continuous function on $[a, b]$ then f is of bounded variation on [a, b].

Unit IV

8. (a) The normed space X is complete iff every absolutely summable sequence is summable.

(b) Let E be a measurable set with $m(E) < \infty$. Then $L^{\infty}(E) \subset L^{p}(E)$ for each p with $1 \leq p < \infty$. Furthermore, if $f \in L^{\infty}(E)$, then $\| f \|_{\infty} = \lim_{p \to \infty} \| f \|_{p}$.

Moreover, explain with an example that the above result is not true for $m(E) = 0$.

9. (a) State and prove Riesz representation theorem.

(b) Prove that the normed space L^{p} $(1 \leq p < \infty)$ is complete.