Roll No. .... **Exam Code : N-21**

Subject Code—11201

M.Sc. EXAMINATION

(Main & Re-appear)

(For Batch 2019 Onwards)

(Second Semester)

MATHEMATICS

MAL-522

Measure and Integration Theory

Time : 3 *Hours* *Maximum Marks* : 70

Note : Attempt *Five* questions in all, selecting *one* question from each Unit. Q. No. **1** is compulsory. All questions carry equal marks.

1. (a) What is measurable function ? If f is a measurable function on a measurable set E, then prove that $f^{-1}(\infty)$ and $f^{-1}(-\infty)$ are measurable sets.

(b) If f and g are measurable function on a measurable set E, then the functions $f+g$ and $\frac{1}{g}(g \neq 0)$ are measurable.

(c) Is the function $f:[0, 1] \to \mathrm{R}$ defined by $f(x)=\sqrt{x}$ Lebesgue Integrable.

(d) Let ϕ and ψ be non-negative simple functions defined on a set E, then prove that $\int_{\mathrm{E}}(\phi+\psi)=\int_{\mathrm{E}}\phi+\int_{\mathrm{E}}\psi$.

(e) Explain Dini's four derivatives. Compute Dini's four derivatives for $f:\mathrm{R} \to \mathrm{R}$ defined by :

$$f(x)=\begin{cases} x\sin\frac{1}{x} & \text{if } x>0 \\ 0 & \text{if } x=0 \\ x+\sqrt{-x\sin^2(\log|x|)} & \text{if } x<0 \end{cases}$$

(f) Let $f \in \mathrm{L}^{\infty}(\mathrm{E})$, then prove that $|f(x)| \leq \|f\|_{\infty}$ a.e. on E.

(g) Let $0 < q < p \leq \infty$. Prove that $L^p \subset L^q$ and there exists a constant $K > 0$ such that $\|f\|_q \leq K\|f\|_p$, for all $f \in L^p$.

Unit I

2. (a) Let E be a measurable set with $m(E) < \infty$. If $\{f_n\}$ be a sequence of measurable functions on E and f be a measurable function such that $f_n(x) \to f$ on E then given $\varepsilon > 0$ and $\delta > 0$ there is a measurable set $A \subset E$ with $m(A) < \delta$ and an integer N such that :

$$|f_n(x) - f(x)| < \epsilon$$

for all $x \in E - A$ and $n \geq N$.

(b) State and prove Egoroff's theorem.

3. (a) If $\{f_n\}$ is a sequence of measurable functions which is fundamental in measure, then there exists a measurable function f such that $\{f_n\}$ converges in measure to f.

(b) State and prove Lusin theorem.

Unit II

4. A bounded function f defined on a measurable set E of finite measure is Lebesgue integrable iff f is measurable.

5. (a) Let f be a non-negative function which is integrable over a set E. Then given $\varepsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $m(A) < \delta$, we have

$$\int_A f < \varepsilon.$$

(b) State and prove Lebesgue dominated convergence theorem.

Unit III

6. (a) Let f be an increasing real valued function on $[a, b]$, then prove that f is differentiable a.e. and the derivative f' is measurable. Furthermore :

$$\int_a^b f'(x)\,dx \le f(b) - f(a).$$

(b) A function f defined on $[a, b]$ is of bounded variations iff it can be expressed as a difference of two monotonic increasing functions on $[a, b]$.

7. (a) Let f be an integrable function defined on $[a, b]$. If $F(x) = \int_a^x f(t)dt + F(a)$, then $F'(x) = f(x)$ a.e. on $[a, b]$.

(b) If f is a absolutely continuous function on $[a, b]$ and $f' = 0$ a.e. on $[a, b]$, then f is constant function on $[a, b]$.

Unit IV

8. (a) State and prove Riesz-Minkowski's inequality for $1 \leq p \leq \infty$.

(b) Let $\{f_n\}$ be a sequence in L^p, $1 \leq p < \infty$ such that $f_n \to f$ a.e. and that $f \in L^p$. If $\lim_{n \to \infty} \| f_n \|_p = \| f \|_p$, then prove $\lim_{n \to \infty} \| f_n - f \|_p = 0$.

9. (a) Let p and q $(1 \leq p, q \leq \infty)$ be two conjugate exponents and $g \in L^p$, then the linear functional defined by $F_g(f) = \int fg$ is a bounded linear functional on L^p such that $\| F_g \| = \| g \|_q$.

(b) Prove that the normed space $L^p (1 \leq p \leq \infty)$ is complete.