Roll No. 22006010047. .... Exam Code : M-23

Subject Code—12002

M. Sc. EXAMINATION

(Batch 2016 Onwards)

(Second Semester)

MATHEMATICS

MAL-522

Measure and Integration Theory

Time : 3 Hours *Maximum Marks : 70*

Note : Attempt *Five* questions in all, selecting *one* question from each Unit. Q. No. **1** is compulsory. All questions carry equal marks.

1. (a) Let f and g be two functions defined on a common domain E such that $f = g$ a.e. and f is measurable then g is also measurable.

(b) Let f and g be bounded measurable functions defined on a set E then prove that :

$$\int_E (f+g) = \int_E f + \int_E g.$$

(c) Let f be a bounded function defined on $[a, b]$. If f is Riemann integrable on $[a, b]$, then it is Lebesgue integrable. Is converse true, if not, explain with an example.

(d) Evaluate the Lebesgue integral of function $f:[0,1] \to R$:

$$f(x) = \begin{cases} \dfrac{1}{x^{1/3}} & 0 < x \le 1 \\ 0 & x = 0 \end{cases}.$$

(e) Explain vitali cover of a set with an example.

(f) Let $0 < q < p \le \infty$, then $L^p \subset L^q$ and there exists a constant $K > 0$ such that : $\| f \|_q \le K \| f \|_p$ for $f \in L^p$.

(g) Define Classical L^p-space. Let $E = (0, \infty)$ and the function $f : E \to R$. defined by $f(x) = \dfrac{1}{\sqrt{(1+x)}}$, then $f \in L^p(E)$ for each $p, 2 < p < \infty$.

Unit I

2\. (a) Define measurable function. Prove that a continuous function defined on a measurable set is measurable. Is converse true. If not, explain with example.

(b) If f is a measurable real valued function on a measurable set E and g be a function defined and continuous on a range of f then gof is a measurable function on E.

3\. (a) State and prove Lusin theorem.

(b) Define convergence in measure. State and prove F. Riesz theorem for convergence in measure.

Unit II

4. (a) A bounded function f defined on a measurable set E of finite measure is Lebesgue integrable iff f is measurable.

(b) State and prove Bounded Convergence theorem.

5. (a) Show that the function $f:[0, \infty[\to \mathrm{R}$, defined by :

$$f(x)=\begin{cases}\frac{\sin(x)}{x} & x \neq 0 \\ 0 & x=0\end{cases}$$

is not Lebesgue integrable over $[0, \infty[$.

(b) State and prove Lebesgue dominated convergence theorem.

Unit III

6. (a) Let f be an increasing real valued function on $[a, b]$, then prove that f is

differentiable a.e. and the derivative f' is measurable. Furthermore :

$$\int_a^b f'(x)dx \leq f(b)-f(a).$$

(b) Define the Dini's four derivatives. Prove that if f is any function on an interval I then $\overline{D}f$ and $D_{-} f$, if exists, are measurable function.

7. (a) Let f be an integrable function on $[a, b]$. If $\int_a^x f(t)dt = 0$, for all $x \in [a, b]$, then $f(x) = 0$ a.e. on $[a, b]$.

(b) If f is a absolutely continuous function on $[a, b]$ then f is of bounded variation on $[a, b]$.

Unit IV

8. (a) State and prove Riesz-Holder inequality for $0 < p < 1$.

(b) Let $\{f_n\}$ be a sequence in L^p, $1 \leq p < \infty$ such that $f_n \to f$ a.e. and that $f \in L^p$. If $\lim_{n\to\infty} \| f_n \|_p = \| f \|_p$ then prove $\lim_{n\to\infty} \| f_n - f \|_p = 0$.

9. (a) Let p and q $(1 \le p, q \le \infty)$ be two conjugate exponents and $g \in L^p$ then the linear functional defined by :

$$F_g(f) = \int fg$$

is a bounded linear functional on L^p such that $\| F_g \| = \| g \|_q$.

(b) Prove that the normed space $L^p (1 \le p \le \infty)$ is complete.