Roll No. 25211061000 4 Exam Code : D-25

Subject Code—26116

M.Sc. (Mathematics) (Under NEP-2020)

EXAMINATION

(Batch 2025 Onwards)

(First Semester)

REAL ANALYSIS

U25MAT102T

Time : 3 Hours *Maximum Marks : 70*

Note : Attempt *Five* questions in all, selecting *one* question from each Unit. Q. No. **1** is compulsory. All questions carry equal marks.

1. (a) If $f \in R(\alpha)$ then $cf \in R(\alpha)$, for every constant c. Also, $\int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha$.

   (b) Evaluate $\int_0^1 x\, dx^2$ using definition of Riemann-Stieltjes integral.

(c) State Dini's theorem.

(d) Give an example of a sequence of continuous functions which has a discontinuous limit function ?

(e) If $A \in L(R^n, R^m)$ and $x \in R^n$, then prove that $A' = A$.

(f) Let
$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^2 + y^2}, & x, y \neq 0 \\ 0 & , \quad x = y = 0 \end{cases}.$$

Prove that $f_{xy} \neq f_{yx}$ at $x = y = 0$.

(g) A bounded monotone function is a function of bounded variation.

Unit I

2. (a) Define Riemann-Stieltjes sum $S(P, f, \alpha)$ on $[a, b]$. If the $\lim_{\|P\| \to 0} S(P, f, \alpha)$ exists than $f \in R(\alpha)$ and $\int_a^b f \, d\alpha = \lim_{\|P\| \to 0} S(P, f, \alpha)$.

(b) Suppose $C_n \geq 0$ for $n = 1, 2, 3, \ldots$ and $\sum_{n=1}^{\infty} C_n$ converges, $<S_n>$ is a sequence of distinct point in (a, b) and

$$\alpha(x) = \sum_{n=1}^{\infty} C_n I(x - S_n).$$

If f be continuous on $[a, b]$, then

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} C_n f(S_n).$$

3. (a) Let $f \in \mathbb{R}$ on $[a, b]$ and $F(x) = \int_a^x f(t)dt, a \leq x \leq b$. Then F is continuous on $[a, b]$. Furthermore, if f is continuous at a point x_0 of $[a, b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

(b) Let $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a, b]$ then :

(i) $fg \in R(\alpha)$

P.T.O.

(ii) $|f| \in R(\alpha)$

(iii) $\left|\int_a^b f \, d\alpha\right| \leq \int_a^b (f) \, d\alpha$.

Unit II

4. (a) Discuss the uniform convergence of the following series :

$$\frac{x^2}{1+x}+\left(\frac{2x^3}{1+2x}-\frac{x^2}{1+x}\right)+\ldots.+\left(\frac{nx^2}{1+nx}-\frac{(n-1)x^2}{1+(n-1)x}\right)+\ldots..$$

on $[0, 1]$.

(b) State Weierstrass M-test. Discuss the uniform convergence of the series :

$$\sum \frac{x}{(n+x^2)^2}.$$

5. (a) Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in R(\alpha)$ on $[a, b]$ for $n = 1, 2, 3, \ldots\ldots$ and suppose $f_n \to f$ uniformly on $[a, b]$. Then $f \in R(\alpha)$ on $[a, b]$ and $\int_a^b f\, d\alpha = \lim_{n\to\infty} \int_a^b f_n\, d\alpha$.

(b) Prove that there exists a real continuous function on the real line which is no where differentiable.

Unit III

6. (a) (i) Let $A \in L(R^n, R^m)$ then $\|A\| < \infty$ then A is uniformly continuous mapping.

(ii) If $A, B \in L(R^n, R^m)$ then $\|A + B\| \leq \|A\| + \|B\|$ and $\|CA\| = |\subseteq|\ \|A\|$, where $\subseteq$ is any scalar.

(iii) If $A \in L(R^n, R^m)$ and $B \in L(R^m, R^k)$ then $\|BA\| \leq \|B\|\ \|R\|$.

(b) Let E be an open set in R^n, f maps E into R^m, f is differentiable at $x_0 \in E$, g maps on open set containing $f(E)$ into R^k and g be differentiable at $f(x_0)$. Then the mapping $F : E \xrightarrow{\text{into}} R^k$ defined by $F(x) = g(f(x))$ is differentiable at x_0 and $F'(x_0) = g'(f(x_0))f'(x_0)$.

7. (a) If f maps an open set $E \subset R^n$ into R^m, then $f \in C'(E)$ if and only if the partial derivatives $D_j f_i$ exists and are continuous on E.

(b) Let f maps a convex open set $E \subset R^n$ into R^m, f be differentiable in E and there be a real no. M such that $\| f'(x) \| \leq M \; \forall \, x \in E$, then prove that $| f(b) - f(a) | \leq M | b - a | \; \forall a, b \in E$.

Unit IV

8. (a) If $f, g \in Bv[a,b]$ and λ is any constant then $f + g$, fg, λf and $\frac{f}{g}$, provided $|g(x)| \geq k$ on $[a, b]$ are also functions of bounded variation on $[a, b]$.

(b) Discuss that the function $f(x) = x \sin x$ on $\left[0, \frac{\pi}{2}\right]$ is a function of bounded variation.

9. (a) Let f be an integrable function on $[a, b]$, then the indefinite integral $F(x) = \int_a^b f(t)dt + c$ is a continuous function of bounded variation on $[a, b]$.

(b) Define vector valued function of bounded variation. Prove that $f : [a,b] \to R^m$ is a function of bounded variation on $[a, b]$ iff each component function f_j is of bounded variation on $[a, b]$.

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